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The effects of variable diffusivity on the development of travelling waves in a class of reaction–diffusion equations

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Contents

In this paper we examine the effects of concentration dependent diffusivity on a reaction–diffusion process which has applications in chemical kinetics and ecology. We consider piecewise classical solutions to an initial boundary-value problem. The existence of a family of permanent form travelling wave solutions is established and the development of the solution of the initial boundary-value problem to the travelling wave of minimum propagation speed is considered. For certain types of initial data, 'waiting time' phenomena are encountered.

1. Introduction

The processes of reaction and diffusion play an important role in a wide variety of chemical, biological and physical systems. A prominent feature associated with the

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concurrent occurrence of reaction and diffusion is the appearance of some type of permanent form wave propagation. This occurs on a faster timescale than any purely diffusive mechanisms and as such can be clearly visible in laboratory experiments. A classical chemical example of this is the Belousov–Zhabotinskii reaction (Zhabotinskii 1964) in which citric acid is oxidized by bromate ions. After an initial induction period, in which diffusion takes place, a well-defined wave-front is observed to propagate as a gross colour change in the reacting mixture. Autocatalytic chemical systems, such as the iodate–arsenous acid scheme (Hanna *et al.* 1982) exhibit travelling wave phenomena and have been extensively studied over the last decade (see, for example, Needham & Merkin 1991). The spread of an advantageous gene through a population (Fisher 1937) is a classical study of reaction and diffusion in theoretical biology. A rather more easily observed phenomenon combining reaction and diffusion is the change that occurs on the surface of a mammalian egg after fertilization. The reaction spreads from the point of contact of the spermatozoon and envelops the egg so as to inhibit further contact by other spermatozoa. A common feature of many of the mathematical models of the above phenomena is the assumption of a constant diffusivity between the reacting elements although the reaction kinetics differ widely. Further details and more extensive surveys of reaction–diffusion models in chemistry and biology are to be found in Winfree (1980), Murray (1989), Jones & Sleeman (1983) and Gray & Scott (1990).

This paper is a study of reaction–diffusion equations with variable diffusivity and the work is presented in terms of general diffusivity and reaction functions with as few restrictions as is possible on the specific functional forms of these. To motivate further the work of this paper and to explain the restrictions on functional form that are necessary, it is instructive to consider briefly a physical model for the flow of a reacting gas through a porous medium. For simplicity we consider a one-dimensional flow with speed $v(x,t)$, density $\rho(x,t)$ and reaction rate $r(x,t)$. The mass continuity equation for such a flow is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v) = r. \tag{1.1}$$

The flow of gas in a porous medium is modelled using d'Arcy's law which relates the flow speed to the pressure gradient, $\partial p/\partial x$, in the form $v = -K\,\partial p/\partial x$, where K is a material constant. If we further assume an ideal gas law in the form $p = k\rho^{\gamma}$, where γ is the ratio of specific heats and k is a scale or reference constant, then variations in the gas density are solutions to the equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(K\gamma k\rho^{\gamma}\frac{\partial \rho}{\partial x}\right)+r. \tag{1.2}$$

The form of reaction function r requires further assumptions about the detailed chemistry of the gas/porous matrix but generally one is led to a form for r which depends only on ρ and has two zeros. These zeros reflect the vanishing of the reaction at zero gas concentration and at a saturation level. It is also worth noting that variable diffusivity reaction–diffusion equations occur in the modelling of insect and microbe populations. The variable diffusivity in these cases arises from population pressure and chemotaxis respectively, whilst the zeros in the reaction function are associated with a logistic model of population behaviour (Murray 1989).

In may ecological models, theory has led to equations of the form

$$u_t = [F(u)]_{xx}+r(u), \tag{1.3}$$

where u is a population density, with $F'(0) = 0$ and $F'(u) > 0$ for $u > 0$, whereas $r(u)$ has logistic type structure. The form of the nonlinear diffusion function $F(u)$ is based on the assumption that individuals in populations disperse to avoid crowding. Discussion of such models may be found in Aronson (1984), Grindrod (1988), Grindrod & Sleeman (1987), Gurtin & MacCamy (1977) and Okubo (1980). In both of these applications, (1.2) and (1.3) have a diffusivity which is low at low density, vanishes at zero density and increases with increasing density. It is diffusivities of this structure which we address in the present paper.

Equations (1.2) and (1.3) are parabolic in nature although they attain a degenerate form where the diffusivity vanishes. In general, initial boundary-value problems for such equations are well posed; however, the solution is not necessarily classical due to the vanishing diffusivity at zero concentration u. It is possible to have (from initial data with compact support) solutions whose support remains finite for all finite $t > 0$, which have a moving boundary at the edge of the support domain where $u = 0$ and the solution is not smooth (see, for example, Lacey *et al.* 1982). The determination of this moving boundary requires an extra condition to those usually placed on parabolic equations. This condition will be seen to arise naturally when an integral conservation form of equations such as (1.2) and (1.3) is considered.

In Lacey *et al.* (1982), weak solutions are considered to the nonlinear diffusion equation $u_t = (u^n u_x)_x$ in $-\infty < x < \infty$, $t > 0$ $(n > 0)$ which have semi-infinite support $(-\infty < x < s(t))$ for all finite $t > 0$. In particular, a class of similarity solutions are obtained, which have finite support and exhibit waiting time phenomena. In this paper we consider a similar initial value problem in a reaction–diffusion context. Solutions with finite support develop, and for particular classes of initial data, waiting times are exhibited. However, with the inclusion of reaction terms, the existence of a family of permanent form travelling waves is established, only one of which is non-classical. It is established that this non-classical travelling wave develops from the initial value problem in the long time. Moreover, the contraction onto this travelling wave solution is rapid, through terms exponential in t as $t \to \infty$. This should be compared with the development of travelling waves in reaction–diffusion equations with constant diffusivity and logistic type kinetics, when the contraction onto the travelling wave of minimum speed is through terms only algebraically small in t as $t \to \infty$ (see, for example, McKean 1975; Bramson 1978; Merkin & Needham 1989; Billingham & Needham 1992).

In the rest of this paper a general theory for variable diffusivity reaction–diffusion equations is developed through the detailed study of the solution to an initial boundary-value problem. A derivation of appropriate conditions at the moving boundary of a non-classical solution together with some general properties of the solution are discussed in §§2 and 3. In §4 we examine the existence of permanent form travelling wave solutions, whereas §5 considers the convergence of solutions of the initial boundary-value problem to a permanent form travelling wave in the long time. Section 6 considers the small time development of the solution, with particular attention to the development of the edge of the support domain, and the existence of waiting time solutions. In §§7 and 8 we examine the waiting time solutions in more detail. Section 9 develops numerical solutions to the initial boundary-value problem. Finally, §10 examines the long time structures of the solution as it approaches the permanent form travelling wave.

2. Conservation laws and differential equations

We consider a scalar reaction–diffusion process in one space dimension for the variable u, which we may regard as the concentration of an autocatalytic chemical species or the density of a population in ecological applications. Under reaction, u reproduces itself at a rate $\hat{R}(u)$ and diffuses (in an unstirred environment) at a rate equal to the gradient of the flux function $\hat{F}(u)$. Here, both $\hat{F}$ and $\hat{R}$ are smooth functions of u. In dimensionless form, the integral conservation law governing the reaction and diffusion of u is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u\,\mathrm{d}x = [F(u)_x]_{x_1}^{x_2} + \int_{x_1}^{x_2} R(u)\,\mathrm{d}x, \tag{2.1}$$

for any $x_2 > x_1 \geqslant 0$, $t > 0$. Here $x > 0$ is the spatial coordinate and $t > 0$ is time. The coordinate x has been made dimensionless on the diffusion length scale associated with the chemical timescale, whereas t has been non-dimensionalized on the chemical timescale. The variable u has been scaled with an equilibrium concentration $u_\mathrm{e} > 0$, with F and R being the dimensionless forms of $\hat{F}$ and $\hat{R}$ respectively.

We restrict attention to the situation when $D(u) \equiv F'(u)$ has $D(0) = 0$ and $D'(0) > 0$, with $D(u)$ monotonically increasing in $u > 0$. These qualitative conditions are typical requirements that arise in ecological and thermal models, where u represents a population density or temperature respectively. In particular, the non-dimensionalization allows us to set $D'(0) = 1$, after which we have

$$D(u) \sim u \quad \text{as} \quad u \to 0^+. \tag{2.2}$$

The reaction function $R(u)$ is taken to have two zeros in $u \geqslant 0$, at $u = 0$ (the unreacted state) and $u = 1$ (the fully reacted state), with $R(u) > 0$ for $u \in (0,1)$ and $R(u) < 0$ for $u \in (1,\infty)$. The equilibrium states $u = 0, 1$ are non-degenerate, so that $R'(0) > 0$ and $R'(1) < 0$. Again the non-dimensionalization allows us to set $R'(0) = 1$, so that

$$\left.\begin{aligned} R(u) &\sim u \quad \text{as} \quad u \to 0^+, \\ R(u) &\sim R'(1)\,(u-1) \quad \text{as} \quad u \to 1. \end{aligned}\right\} \tag{2.3a}$$

In addition, we include the condition

$$R(u) \leqslant u \quad \forall u \geqslant 0, \tag{2.3b}$$

which is a technical condition that will be required at a later stage.

We examine the initial boundary-value problem that arises when a localized quantity of u is introduced initially into the otherwise unreacting state $u \equiv 0$. Under these circumstances, equation (2.1) must be solved in $x, t > 0$ subject to the following initial and boundary conditions:

$$u(x,0) = \begin{cases} u_0\,g(x), & 0 \leqslant x \leqslant \sigma, \\ 0, & x > \sigma, \end{cases} \tag{2.4}$$

$$u_x(0,t) = 0, \quad t > 0, \tag{2.5}$$

$$u(x,t) \to 0 \quad \text{as} \quad x \to \infty, \quad t > 0. \tag{2.6}$$

The function $g(x)$ is positive, monotonically decreasing and analytic in $0 \leqslant x \leqslant \sigma$, with, in particular $\max_{0 \leqslant x \leqslant \sigma}\{g(x)\} = 1$, $0 < u_0 < 1$ and,

$$g(x) \sim u_0^{-1} g_m (x-\sigma)^m \quad \text{as} \quad x \to \sigma^- \tag{2.7}$$

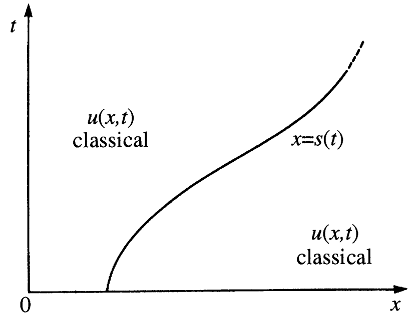

Figure 1. The x, t domain. $u(x,t)$ has derivative jumps across $x = s(t)$.

for some non-zero constant g_m, $m = 0, 1, 2, \ldots$. The parameter σ defines the support of the initial data $g(x)$.

With $u(x,t)$ suitably differentiable in $x, t > 0$, the integral conservation law (2.1) becomes

$$\frac{\partial u}{\partial t} = D(u)\frac{\partial^2 u}{\partial x^2} + D'(u)\left(\frac{\partial u}{\partial x}\right)^2 + R(u), \quad x, t > 0, \tag{2.8}$$

and the initial boundary-value problem reduces to solving (2.8) in $x, t > 0$ subject to conditions (2.4)–(2.6). However, we observe that this initial boundary-value problem is singular since $D(0) = 0$, and the initial data (2.4) has $u \equiv 0$ for $x > \sigma$. Under these circumstances we expect that a classical solution to (2.8), (2.4)–(2.6) will not exist globally, nor even locally. To avoid this situation, we return to the original integral conservation law, (2.1), which admits a much broader class of solutions than its differential form (2.8). In particular, we consider solutions $u(x,t)$ to the initial boundary-value problem (2.1), (2.4)–(2.6) on

$$D_T = \{(x,t) \in \mathbb{R}^2 : 0 < x < \infty, \quad 0 < t \leqslant T\},$$

which have $u(x,t)$ continuous everywhere on D_T, whilst u_t, u_x, u_{xx} exist and are continuous throughout D_T *except* possibly along a finite number of continuous and piecewise-differentiable curves in D_T. However, the limits of u_t, u_x, u_{xx} exist as points on such curves are approached from either side. We denote this class of functions on D_T as $C_p[D_T]$, and refer to this as the class of piecewise classical solutions to (2.1), (2.4)–(2.6) on D_T. Clearly, all the operations in (2.1) are well defined for $u \in C_p[D_T]$.

(*a*) *Piecewise classical solutions*

Let $u(x,t)$ be a piecewise classical solution of (2.1), (2.4)–(2.6) on D_T, with a derivative jump across the smooth curve $x = s(t)$, $0 \leqslant t \leqslant T$, as depicted in figure 1. We now consider the conditions which must be satisfied across $x = s(t)$ by $u(x,t)$ and its derivatives. On either side of $x = s(t)$ ($0 \leqslant x < s(t)$, $x > s(t)$), $u(x,t)$ is a classical solution, and thus satisfies the differential form of (2.1), namely (2.8). Across $x = s(t)$, (2.1) is satisfied. We next take $0 < x_1 < s(t) < x_2$ in (2.1), and re-write it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{x_1}^{s(t)} u\,\mathrm{d}x + \int_{s(t)}^{x_2} u\,\mathrm{d}x\right\} = [F(u)_x]_{x_1}^{x_2} + \int_{x_1}^{s(t)} R(u)\,\mathrm{d}x + \int_{s(t)}^{x_2} R(u)\,\mathrm{d}x. \tag{2.9}$$

With $u \in C_p[D_T]$, we may take the limits $x_1 \to s(t)^-$, $x_2 \to s(t)^+$ in (2.9), which leads to

$$D(u(s(t),t))\,[u_x(s(t)^+,t) - u_x(s(t)^-,t)] = 0, \tag{2.10}$$

for $0 < t \leqslant T$. By hypothesis, $u_x(s(t)^+, t) \neq u_x(s(t)^-, t)$ (otherwise the solution is classical) and so (2.10) dictates that $u(s(t), t)$ must be a zero of the diffusivity function $D(u)$. However $D(u)$ has only the single zero at $u = 0$, and so we have

$$U(t) = u(s(t),\, t) \equiv 0, \quad 0 < t \leqslant T. \tag{2.11}$$

Moreover, since $s(t)$ is smooth and $u \in C_{\mathrm{p}}[D_T]$, we have, from (2.11),

$$\lim_{x \to s(t)^+} \left[\frac{\partial u}{\partial t} + \dot{s}(t)\frac{\partial u}{\partial x}\right] = \lim_{x \to s(t)^-} \left[\frac{\partial u}{\partial t} + \dot{s}(t)\frac{\partial u}{\partial x}\right] = \dot{U}(t) \equiv 0 \tag{2.12}$$

for $0 < t \leqslant T$. Now, $u(x,t)$ satisfies (2.8) in $x < s(t)$ and $x > s(t)$ with finite limits in all derivatives as $x \to s(t)^{\pm}$. Therefore, using (2.8) in (2.12) we have

$$\lim_{x \to s(t)^+} [D(u)\, u_{xx} + D'(u)\, u_x^2 + \dot{s}(t)\, u_x + R(u)]$$

$$= \lim_{x \to s(t)^-} [D(u)\, u_{xx} + D'(u)\, u_x^2 + \dot{s}(t)\, u_x + R(u)] \equiv 0 \tag{2.13}$$

for $0 < t \leqslant T$. On using (2.2) and (2.11) in (2.13) we arrive at the additional regularity conditions,

$$u_x^+(u_x^+ + \dot{s}(t)) = u_x^-(u_x^- + \dot{s}(t)) = 0, \quad 0 < t \leqslant T, \tag{2.14}$$

where u_x^+ and u_x^- are the limits of u_x as $x \to s(t)$ from above and below respectively. As $u_x^+ \neq u_x^-$, then (2.14) reduces to

$$u_x^- = -\dot{s}(t), \quad u_x^+ = 0, \tag{2.15a}$$

or

$$u_x^- = 0, \quad u_x^+ = -\dot{s}(t). \tag{2.15b}$$

Thus a solution of the integral conservation law (2.1) in $C_{\mathrm{p}}(D_T)$ satisfies equation (2.8) throughout D_T, except possibly across a finite number of smooth curves, $x = s(t)$, in D_T, across which conditions (2.11) and (2.15a) or (2.15b) must be satisfied. The remainder of the paper is concerned with constructing a piecewise classical solution to the initial boundary-value problem (2.1), (2.4)–(2.6).

(b) *Reformulation of the initial boundary-value problem*

The form of the initial data in (2.4) leads us to consider a solution of the initial boundary-value problem (2.1), (2.4)–(2.6) which is piecewise classical, with a single jump across $x = s(t)$, at which conditions (2.11) and (2.15a) must be satisfied. The initial boundary-value problem can now be reformulated as follows:

$0 \leqslant x \leqslant s(t)$, $t \geqslant 0$

$$u_t = D(u)\, u_{xx} + D'(u)\,(u_x)^2 + R(u), \tag{2.16}$$

$$u(x,0) = u_0\, g(x), \quad s(0) = \sigma, \tag{2.17a, b}$$

$$u_x(0,t) = 0, \tag{2.18}$$

$$u(s(t),t) = 0, \quad u_x(s(t),t) = -\dot{s}(t). \tag{2.19a, b}$$

$x \geqslant s(t), t > 0$

$$u_t = D(u)\, u_{xx} + D'(u)\,(u_x)^2 + R(u), \tag{2.20}$$

$$u(x,0) \equiv 0, \tag{2.21}$$

$$u(x,t) \to 0 \quad \text{as} \quad x \to \infty, \tag{2.22}$$

$$u(s(t),t) = 0, \quad u_x(s(t),t) = 0. \tag{2.23a, b}$$

We observe immediately that the appropriate solution to (2.20)–(2.23*a*, *b*) has

$$u(x,t) \equiv 0, \quad x \geqslant s(t), \quad t > 0, \tag{2.24}$$

and it is shown in the Appendix that this is unique. It remains to consider the initial boundary-value problem (2.16)–(2.19*a*, *b*), which must be solved for $u(x,t)$ and $s(t)$ in $0 \leqslant x \leqslant s(t)$, $t > 0$. We note that (2.16)–(2.19*a*, *b*) is a nonlinear free boundary problem, with conditions (2.19*a*, *b*) being of the Stefan type (see, for example, Friedman 1964). We begin by obtaining some general properties of the solution to (2.16)–(2.19), which we henceforth refer to as IBVP.

3. General properties of IBVP

We first obtain bounds on $u(x,t)$.

Proposition 3.1. *Let $u(x,t)$, $s(t)$ be the solution of* IBVP *on $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$, for any $T > 0$, then $0 \leqslant u(x,t) \leqslant 1$.*

Proof. We first extend the definitions of $D(u)$ and $R(u)$ into $u < 0$, so that

$$D(u) \equiv 0, \quad R(u) = u^2 \quad \text{for} \quad u < 0. \tag{3.2}$$

With (3.2) we note that both $D(u)$ and $R(u)$ are Lipschitz continuous for $-\infty < u < \infty$.

Now let $\epsilon > 0$ be arbitrary and we show that $u(x,t) \geqslant -\epsilon$ on $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$. For $x = s(t)$, $0 \leqslant t \leqslant T$, then $u(s(t),t) \geqslant -\epsilon$ via conditions (2.17*a*) and (2.19*a*). It remains to consider $u(x,t)$ for $0 \leqslant x < s(t)$, $0 \leqslant t \leqslant T$. Clearly we have

$$u(x,0) > -\epsilon, \quad 0 \leqslant x < \sigma \tag{3.3}$$

via condition (2.17*a*) and the properties of $g(x)$. Now suppose there are $0 < t^* \leqslant T$ and $0 \leqslant x^* < s(t^*)$ with $u(x^*,t^*) < -\epsilon$. Then, via (3.3), there exist $0 < t^{\mathrm{c}} < t^*$ and $0 \leqslant x^{\mathrm{c}} < s(t^{\mathrm{c}})$, with

$$u(x^{\mathrm{c}},t^{\mathrm{c}}) = -\epsilon, \quad u_x(x^{\mathrm{c}},t^{\mathrm{c}}) = 0, \quad u_{xx}(x^{\mathrm{c}},t^{\mathrm{c}}) \geqslant 0, \tag{3.4}$$

$$u_t(x^{\mathrm{c}},t^{\mathrm{c}}) \leqslant 0. \tag{3.5}$$

However, equation (2.16) holds at $x = x^{\mathrm{c}}$, $t = t^{\mathrm{c}}$, leading to

$$u_t(x^{\mathrm{c}},t^{\mathrm{c}}) = D(-\epsilon)\, u_{xx}(x^{\mathrm{c}},t^{\mathrm{c}}) + D'(-\epsilon)\,[u_x(x^{\mathrm{c}},t^{\mathrm{c}})]^2 + R(-\epsilon),$$

which gives, via (3.4) and (3.2),

$$u_t(x^{\mathrm{c}},t^{\mathrm{c}}) = \epsilon^2 > 0,$$

contradicting (3.5). We conclude that $u(x,t) \geqslant -\epsilon$ on $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$ for any $\epsilon > 0$, and the left-hand inequality follows directly.

A similar approach also establishes the right-hand inequality, on recalling that $0 < u_0 < 1$. □

An immediate consequence of proposition (3.1), which follows after use of condition (2.19*b*) and the mean value theorem is that

$$\dot{s}(t) \geqslant 0, \quad 0 \leqslant t \leqslant T, \tag{3.6}$$

which implies that $s(t)$ is non-decreasing.

4. Permanent form travelling waves

We expect that the long time development of IBVP may involve the propagation of a travelling wave of permanent form in $x > 0$, separating the unreacted state, $u \equiv 0$, ahead, from the fully reacted state, $u \equiv 1$, to the rear. Therefore, before developing IBVP further, we examine the possible class of piecewise classical permanent form travelling waves that can be sustained by the integral conservation law (2.1). We note at this stage that permanent form travelling wave solutions to reaction–diffusion equations of the type (2.8) with $D(u) \equiv u^n$, $n \in \mathbb{N}$, and $R(u) \equiv u(1-u)$ have been discussed previously by Newman (1980) and reviewed recently by Murray (1989) and Grindrod (1991). This section generalizes these results to the broader class of functions $D(u)$ and $R(u)$ defined in §2.

To proceed, we make the following definition.

4.1. *Definition.* A permanent form travelling wave solution of the integral conservation law (2.1) is a non-negative solution which depends only on the single variable $z \equiv x - \gamma(t)$ (where $\gamma(t)$ is the position of the wave-front), and satisfies the conditions $u \to 0$ as $z \to \infty$, $u \to 1$ as $z \to -\infty$. In addition the solution should be continuous and piecewise classical for $-\infty < z < \infty$.

A permanent form travelling wave therefore satisfies the differential equation

$$D(u)\,u_{zz} + D'(u)\,u_z^2 + vu_z + R(u) = 0, \tag{4.2}$$

at all but a finite number of points $z = z_i$ $(i = 1, 2, \dots, N)$, at which the appropriate form of conditions (2.11), (2.15*a*) or (2.15*b*) must hold, namely,

$$u(z_i) = 0, \tag{4.3}$$

$$u_z^- = -v, \quad u_z^+ = 0, \quad \text{or} \quad u_z^- = 0, \quad u_z^+ = -v, \tag{4.4}$$

with the usual $\pm$ notation. In the above $v = \dot{\gamma}(t)$. However, since u is a function of z alone, (4.2) determines that the wave-front propagation speed v must be constant. Moreover the symmetry of (4.2)–(4.4) implies that we need only consider the situation when $v > 0$. The boundary conditions which must be satisfied by $u(z)$ are

$$u(z) \to 0 \quad \text{as} \quad z \to \infty, \tag{4.5a}$$

$$u(z) \to 1 \quad \text{as} \quad z \to -\infty. \tag{4.5b}$$

The problem (4.2)–(4.5) can be thought of as a nonlinear eigenvalue problem with the positive propagation speed, v, being the eigenvalue. We study (4.2)–(4.5) in the phase plane.

(*a*) *The phase plane*

We introduce the variable $\omega = u_z$ and write (4.2) as the equivalent system

$$u_z = \omega, \quad \omega_z = \{-D'(u)\,\omega^2 - v\omega - R(u)\}/D(u). \tag{4.6}$$

The trajectories of system (4.6) in the (u, ω) phase plane satisfy the first order ordinary differential equation

$$\mathrm{d}\omega/\mathrm{d}u = \{-D'(u)\,\omega^2 - v\omega - R(u)\}/\omega D(u), \tag{4.7}$$

from which it is clear that the singular system (4.6) has the same phase portrait as the regular system

$$\bar{u}_z = \bar{\omega} D(\bar{u}), \tag{4.8a}$$

$$\bar{\omega}_z = -D'(\bar{u})\,\bar{\omega}^2 - v\bar{\omega} - R(\bar{u}). \tag{4.8b}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

We are thus able to characterize the phase portrait of the singular system (4.6) via that of the regular system (4.8). The latter has three equilibrium points in the $(\bar{u}, \bar{\omega})$ phase plane, at $(0,0)$, $(0,-v)$, $(1,0)$. Thus a solution of (4.2)–(4.4) which satisfies the conditions (4.5a) and (4.5b) requires the existence of a directed integral path of equations (4.8a) and (4.8b) which connects the equilibrium point $(1,0)$ to $(0,0)$. The path must be regular in $\bar{u} > 0$ but may jump from $(0,-v)$ to $(0,0)$ according to (4.4) (observing that the second of conditions (4.4) is ruled out by definition (4.1) when $v > 0$).

We begin by examining the local behaviour in the neighbourhood of the equilibrium points of (4.8). Linearization of equations (4.8) at the point $(1,0)$ shows that it is a simple saddle point with eigenvalues and associated eigenvectors given by

$$\left.\begin{aligned} \lambda_1 &= -\tfrac{1}{2}\{[v^2 - 4D(1)R'(1)]^{\frac{1}{2}} + v\}, \quad & \boldsymbol{e}_{\lambda_1} &= [D(1), \lambda_1]^{\mathrm{T}}, \\ \lambda_2 &= \tfrac{1}{2}\{[v^2 - 4D(1)R'(1)]^{\frac{1}{2}} - v\}, \quad & \boldsymbol{e}_{\lambda_2} &= [D(1), \lambda_2]^{\mathrm{T}}. \end{aligned}\right\} \tag{4.9}$$

Hence there are two possible integral paths which satisfy (4.5b), namely those two paths which leave the equilibrium point (1, 0) on the unstable manifold. However, the path that is directed from (1, 0) into the quadrant $\bar{u}$, $\bar{\omega} > 0$ must have $\bar{u}$ monotonically increasing and therefore cannot reach (0, 0) or $(0, -v)$, and we must rule this out as it cannot satisfy condition (4.5a). Therefore, a solution to (4.2)–(4.5) must correspond to that integral path of system (4.8) which leaves the equilibrium point $(1,0)$ on that section of the unstable manifold that points into the region $\bar{u} < 1$, $\bar{\omega} < 0$. We label this S_1, and the local behaviour in the neighbourhood of the equilibrium point is shown in figure 2.

Linearization of equations (4.8) about the equilibrium point (0, 0) shows that it is non-simple with eigenvalues and associated eigenvectors given by

$$\left.\begin{aligned} \mu_1 &= -v, \quad & \boldsymbol{e}_{\mu_1} &= [0, 1]^{\mathrm{T}}, \\ \mu_2 &= 0, \quad & \boldsymbol{e}_{\mu_2} &= [1, -1/v]^{\mathrm{T}}. \end{aligned}\right\} \tag{4.10}$$

Hence, the linearized equations do not give a classification of the local behaviour. However, a straightforward application of the centre manifold theorem (see, for example, Carr 1981) shows that the equilibrium point has a unique, one dimensional, invariant stable manifold, locally tangential to $\boldsymbol{e}_{\mu_1}$, and a one dimensional invariant centre manifold locally tangential to $\boldsymbol{e}_{\mu_2}$. Moreover, Carr's theorem (Carr 1981) guarantees that any paths in the vicinity of $(0,0)$, except those in the stable manifold, contract rapidly onto the centre manifold. Thus, the dynamics on the centre manifold determine the nature of the equilibrium point (0, 0). An approximation to the centre manifold is readily obtained as

$$\bar{\omega}_c(\bar{u}) \sim -\frac{1}{v}\bar{u} - \frac{1}{v}\left[\frac{2}{v^2} + \tfrac{1}{2}R''(0)\right]\bar{u}^2 + \ldots \tag{4.11}$$

as $\bar{u} \to 0$. Thus, on the centre manifold we have, via (4.8a), that $\bar{u}_z < 0$ in both $\bar{u} > 0$ and $\bar{u} < 0$. Therefore, all paths starting to the right of the stable manifold enter $(0,0)$ along the centre manifold whereas those starting to the left of the stable manifold are swept away from $(0,0)$ close to the centre manifold. We conclude that $(0,0)$ is a saddle node with the nodal region to the right of the stable manifold and the saddle region to the left. All paths which enter $(0,0)$ do so along the centre manifold (4.11) *except* the two paths that form the stable manifold. These enter $(0,0)$ along the $\bar{\omega}$-axis. The phase portrait in the neighbourhood of $(0,0)$ is shown in figure 2.

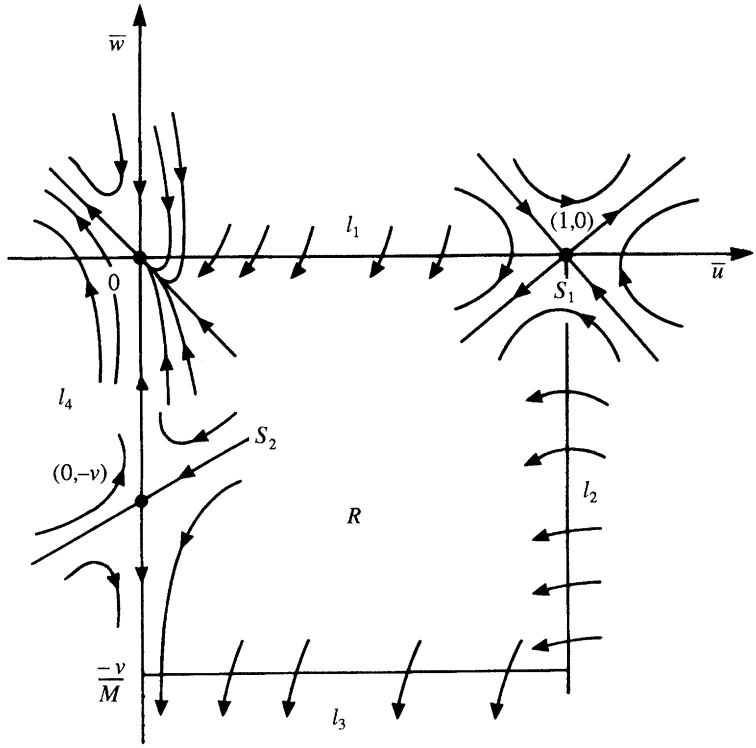

Figure 2. The ($\bar{u}$, $\bar{\omega}$) phase plane.

Finally, linearization about the equilibrium point $(0, -v)$ shows that it is a simple saddle point, with eigenvalues and associated eigenvectors given by

$$\left.\begin{aligned} \nu_1 &= v, \quad \boldsymbol{e}_{\nu_1} = [0, 1]^{\mathrm{T}}, \\ \nu_2 &= -v, \quad \boldsymbol{e}_{\nu_2} = \left[1, \frac{1}{2v} + vD''(0)\right]^{\mathrm{T}}. \end{aligned}\right\} \tag{4.12}$$

The phase portrait in the neighbourhood of $(0, -v)$ is illustrated in figure 2, with the section of the stable manifold in $\bar{u} > 0$ labelled as S_2.

We also note that the whole of the $\bar{\omega}$-axis forms an integral path of the system (4.8), as illustrated in figure 2. Next we define the rectangular region

$$R = \{(\bar{u}, \bar{\omega}) : 0 \leqslant \bar{u} \leqslant 1, \quad -v/M \leqslant \bar{\omega} \leqslant 0\} \tag{4.13}$$

in the $(\bar{u}, \bar{\omega})$ phase plane, with $M = \inf\{D'(\bar{u}) : 0 \leqslant \bar{u} \leqslant 1\}$. We readily observe that integral paths of (4.8) strictly enter R along the edges

$$\left.\begin{aligned} L_1 &= \{(\bar{u}, \bar{v}) : \bar{\omega} = 0, \quad 0 < \bar{u} < 1\}, \\ L_2 &= \{(\bar{u}, \bar{v}) : -v/M < \bar{\omega} < 0, \quad \bar{u} = 1\}, \end{aligned}\right\} \tag{4.14}$$

but cannot enter R through the remaining edges

$$\left.\begin{aligned} L_3 &= \{(\bar{u}, \bar{v}) : \bar{\omega} = -v/M, \quad 0 < \bar{u} < 1\}, \\ L_4 &= \{(\bar{u}, \bar{v}) : -v/M < \bar{\omega} < 0, \quad \bar{u} = 0\}. \end{aligned}\right\} \tag{4.15}$$

The situation is shown in figure 2. It is now straightforward to observe that the stable manifold S_2 of the equilibrium point $(0, -v)$ must enter R either through L_1 or L_2 but cannot enter through L_3 or L_4. Thus there are three cases to consider, namely,

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

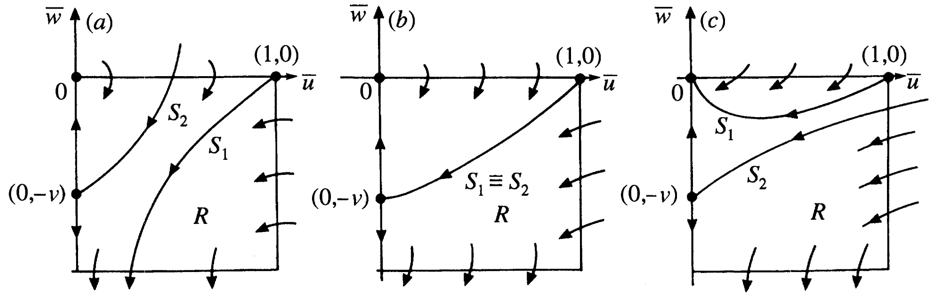

Figure 3. The locations of S_1 and S_2 in the three cases (*a*) case (i), (*b*) case (ii), (*c*) case (iii).

(i) S_2 enters R through L_1. In this case, S_1 lies below S_2 in R and cannot therefore connect to $(0,0)$. In fact S_1 must leave R through L_3. Hence the eigenvalue problem (4.2)–(4.5) has no solution in this case (see figure 3*a*).

(ii) S_2 connects with the equilibrium point $(1,0)$. In this case, in terms of the original system (4.6), this path approaches $(1,0)$ as $z \to -\infty$, but reaches $(0,\,-v)$ as $z \to 0^-$. The jump conditions (4.4) then allow a jump from $(0,\,-v)$ at $z=0^-$ to $(0,0)$ in $z>0$. Therefore in this case there is a unique solution to the eigenvalue problem (4.2)–(4.5). This solution is not classical and has a single jump in u_z at $z=0$. For $z>0$, $u(z)\equiv 0$, whereas for $z<0$, $u(z)$ is monotonically decreasing (see figure 3*b*).

(iii) S_2 enters R through L_2. In this case S_2 lies below S_1 in R. Thus S_1 enters an invariant region in which the only attractor is $(0,0)$ and so S_1 must connect to $(0,0)$. Hence the eigenvalue problem (4.2)–(4.5) has a unique solution (see figure 3*c*). In this case the solution is classical and monotonically decreasing.

It remains to decide which of the above cases holds for each $v>0$. We achieve this by defining a function $f(v)$ to be $\bar{u}_2$, the value of $\bar{u}$ at which S_2 crosses the $\bar{u}$-axis in case (i), and to be $1-\bar{\omega}_2$ in case (iii), where $\bar{\omega}_2$ is the value of $\bar{\omega}$ at which S_2 crosses the line $\bar{u}=1$. In case (ii) we set $f(v)=1$. This gives a well-defined value $f(v)\geqslant 0$ for all $v>0$. Moreover, because the right-hand sides of (4.8) are differentiable functions of the parameter v, as well as $\bar{u}$ and $\bar{\omega}$, the integral paths depend continuously on v (see, for example, Hirsch & Smale 1974), and hence $f(v)$ is a continuous function in $v>0$.

With $f(v)$ defined in this way, we have established the following proposition.

Proposition 4.16. *The eigenvalue problem* (4.2)–(4.5) *has*

(i) *a unique solution for each* $v>0$ *such that* $f(v)\geqslant 1$. *For* $f(v)>1$, *the solution is classical whereas for* $f(v)=1$, *the solution has a single jump in* u_z;

(ii) *no solution for each* $v>0$ *such that* $0<f(v)<1$. □

It remains to compute the function $f(v)$ in $v>0$.

(*b*) *The function* $f(v)$

The following proposition establishes an important property of the function $f(v)$.

Proposition 4.17. *The function* $f(v)$ *is strictly monotonically increasing for* $v>0$.

Proof. From equations (4.8) we have

$$\frac{\partial}{\partial v}\left[\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}\bar{u}}\right] = -\frac{1}{D(\bar{u})} < 0 \tag{4.18}$$

for all $\bar{u} > 0$. Hence, as v increases, the vector $[\bar{u}_z, \bar{\omega}_z]^{\mathrm{T}}$ defined by equations (4.8) rotates clockwise at any given point in R and tends monotonically to the upward vertical as $v \to \infty$. With $v = v_0 > 0$ we define the region

$$R_{v_0} = \{(\bar{u}, \bar{\omega}) : \bar{\omega}_s(\bar{u})|_{v=v_0} \leqslant \bar{\omega} \leqslant 0, \quad 0 \leqslant \bar{u} \leqslant 1\},$$

where $\bar{\omega} = \bar{\omega}_s(\bar{u})|_{v=v_0}$ is the equation describing the integral path S_2 in R at $v = v_0$. When $v = v_1 > v_0$, all integral paths on the part of the boundary of R_{v_0} given by the segment of the curve $\bar{\omega} = \bar{\omega}_s(\bar{\omega})|_{v=v_0}$ in R enter R_{v_0}, by (4.18). All integral paths on the remaining two straight line segments of the boundary of R_{v_0} also cannot leave R_{v_0}, as shown in figure 2. Since at $v = v_1$, S_2 originates at $(0, v_1) \in R \backslash R_{v_0}$, it cannot then enter R_{v_0} with decreasing z, and the result follows via the definition of $f(v)$. □

To determine $f(v)$ in $v > 0$, we must follow the path S_2, given by $\bar{\omega} = \bar{\omega}_s(\bar{u})$ as $\bar{u}$ increases from zero. The integral path $\bar{\omega}_s(\bar{u})$ satisfies the condition

$$\bar{\omega}_s(\bar{u}) \sim -v + [1/(2v) + vD''(0)]\,\bar{u} + \ldots \quad \text{as} \quad \bar{u} \to 0^+ \tag{4.19}$$

and the differential equation

$$\mathrm{d}\bar{\omega}_s/\mathrm{d}\bar{u} = -[D'(\bar{u})\,\bar{\omega}_s^2 - v\bar{\omega}_s - R(\bar{u})]/\bar{\omega}_s D(\bar{u}), \quad \bar{u} > 0, \tag{4.20}$$

via (4.8) and (4.12). We consider the asymptotic forms of $\bar{\omega}_s(\bar{u})$ when $v \ll 1$ and $v \gg 1$.

$0 < v \ll 1$

As $v \to 0$, condition (4.19) suggests that $\bar{\omega}_s = O(v)$, whereas (4.20) gives $\bar{u} = O(v^2)$. We therefore introduce the scaled variables, $\hat{\omega}_s$ and $\hat{u}$, as

$$\bar{u} = v^2 \hat{u}, \quad \bar{\omega}_s = v\hat{\omega}_s, \tag{4.21}$$

with $\hat{u}, \hat{\omega}_s = O(1)$ as $v \to 0$. In terms of the scaled variables (4.21), the initial value problem (4.19), (4.20) becomes

$$\mathrm{d}\hat{\omega}_s/\mathrm{d}\hat{u} = -(\hat{\omega}_s^2 - \hat{\omega}_s - \hat{u})/\hat{u}\hat{\omega}_s, \quad \hat{\omega}_s \sim -1 + \tfrac{1}{2}\hat{u} + \ldots \quad \text{as} \quad \hat{u} \to 0^+, \tag{4.22}$$

at leading order in v. A numerical integration of (4.22) shows that $\hat{\omega}_s(\hat{u})$ crosses the $\hat{u}$-axis when $\hat{u} = u^* \approx 0.8587$. Hence, when $0 < v \ll 1$, case (i) is obtained and $\bar{u}_2 \sim v^2 u^*$, which gives,

$$f(v) \sim u^* v^2 \quad \text{as} \quad v \to 0^+. \tag{4.23}$$

$v \gg 1$

As $v \to \infty$ condition (4.19) suggests that $\bar{\omega}_s = O(v)$ whilst $\bar{u} = O(1)$. We therefore introduce the scaled variable $\tilde{\omega}_s = \bar{\omega}_s v^{-1}$ with $\tilde{\omega}_s = O(1)$ as $v \to \infty$. In terms of the scaled variable the initial value problem (4.19), (4.20) becomes, at leading order as $v \to \infty$,

$$\frac{\mathrm{d}\tilde{\omega}_s}{\mathrm{d}\bar{u}} + \frac{D'(\bar{u})}{D(\bar{u})}\tilde{\omega}_s = -\frac{1}{D(\bar{u})}, \quad \tilde{\omega}_s \to -1 \quad \text{as} \quad \bar{u} \to 0^+. \tag{4.24}$$

The solution to (4.24) is readily obtained as

$$\tilde{\omega}_s(\bar{u}) = -\bar{u}/D(\bar{u}). \tag{4.25}$$

An examination of (4.25) shows that $\tilde{\omega}_s < 0$ for all $0 \leqslant \bar{u} \leqslant 1$, and so case (iii) holds when $v \gg 1$, and $\bar{\omega}_2 \sim -v/D(1)$. Therefore we have,

$$f(v) \sim v/D(1) + 1 \quad \text{as} \quad v \to \infty. \tag{4.26}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$v = O(1)$

On using the asymptotic forms for $0 < v \ll 1$, (4.23) and $v \gg 1$, (4.26), we observe that $f(v) \to 0$ as $v \to 0$ and $f(v) \to \infty$ as $v \to \infty$. Hence, from Proposition 4.17 the equation $f(v) = 1$ has a single positive solution in $v > 0$, at $v = v^*$, say. Also $f(v) > 1$ for $v > v^*$ and $0 < f(v) < 1$ for $0 < v < v^*$. We can use Proposition (4.16) to establish the following theorem.

Theorem 4.27. *The eigenvalue problem* (4.2)–(4.5) *has a solution if and only if* $v \geqslant v^*$, *where* v^* *is the single positive root of* $f(v) = 1$. *Moreover, the solution is unique. For* $v > v^*$ *the solution is classical, whereas for* $v = v^*$ *the solution has a single jump in* u_z.

We now examine properties of the travelling wave solutions.

(*c*) *Properties of the solutions*

Since for each $v \geqslant v^*$, the trajectory representing the travelling wave solution lies in $\omega < 0$, then each travelling wave is monotonically decreasing. For a given $v \geqslant v^*$ the solution of the eigenvalue problem leaves the saddle point at (1, 0) along the unstable manifold S_1, so that $\omega \sim [\lambda_2/D(1)]\,(u-1)$ as $u \to 1^-$, using (4.9). Substitution into (4.6), followed by an integration then gives

$$u(z) \sim 1 - A\,\mathrm{e}^{\lambda_2 z/D(1)} \quad \text{as} \quad z \to -\infty, \tag{4.28}$$

where A is a positive constant. Thus, for every $v \geqslant v^*$, $u(z)$ decays exponentially in z to its final value of unity, as $z \to -\infty$. However, for $v = v^*$, $u(z) \equiv 0$ for $z > 0$, with, from (4.12) and (4.6),

$$u(z) \sim -v^* z \quad \text{as} \quad z \to 0^-, \tag{4.29}$$

whereas for $v > v^*$, from (4.10) and (4.6),

$$u(z) \sim B\,\mathrm{e}^{-z/v} \quad \text{as} \quad z \to \infty, \tag{4.30}$$

with B a positive constant.

We conclude that there is a one parameter family of piecewise classical permanent form travelling wave solutions to the integral conservation law (2.1). These are parametrized by their propagation speed $v \geqslant v^*$. For $v > v^*$ the travelling waves are classical, whereas the minimum speed travelling wave with $v = v^*$ has a single jump in u_z at $z = 0$.

(*d*) *An example: the modified Fisher equation*

As an example, we consider the case in which

$$D(u) \equiv u, \quad R(u) \equiv u(1-u). \tag{4.31}$$

Thus, we are considering equation (2.1) with a reaction function corresponding to the Fisher equation, but with a diffusivity which varies linearly with u.

This equation has been studied previously by Newman (1980) and is discussed by Grindrod (1991). However, for the purposes of the present paper it is instructive to give a brief re-examination. In this case the boundary value problem for the minimum speed travelling wave is

$$u u_{zz} + u_z^2 + v^* u_z + u(1-u) = 0, \quad -\infty < z < 0, \tag{4.32}$$

$$u(0) = 0, \quad u_z(0) = -v^*, \quad u(z) \to 1 \quad \text{as} \quad z \to -\infty. \tag{4.33a–c}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

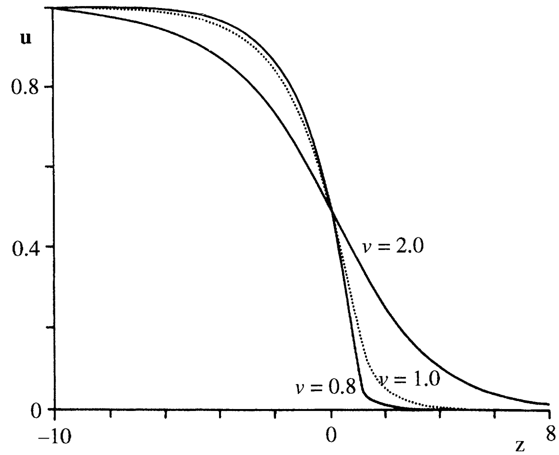

Figure 4. The travelling wave for the modified Fisher equation with $v = 0.8,\ 1.0,\ 2.0 > 1/\surd 2$.

The existence and uniqueness of a solution to this problem is guaranteed by (4.27). In this case a closed form solution can be constructed. Putting $p \equiv u_z$, then (4.32), (4.33) becomes

$$upp_u + p^2 + v^* p + u(1-u) = 0, \quad 0 < u < 1, \tag{4.34}$$

$$p(1) = 0, \quad p(0) = -v^*, \tag{4.35 a, b}$$

which is a Riccati equation. We try a solution of the form $p(u) = \alpha u + \beta$, with α, β constant. On substitution into (4.34) we find that α, β and v^* must be chosen as

$$\alpha = \tfrac{1}{\surd 2}, \quad \beta = -\tfrac{1}{\surd 2}, \quad v^* = \tfrac{1}{\surd 2}, \tag{4.36}$$

after which the solution of (4.34), (4.35) is

$$p(u) = \tfrac{1}{\surd 2}(u-1), \quad v^* = \tfrac{1}{\surd 2}. \tag{4.37}$$

Finally we have, from (4.37),

$$\left.\begin{aligned} u_z &= \tfrac{1}{\surd 2}(u-1), \quad -\infty < z < 0, \\ u(0) &= 0, \quad u(z) \to 1 \quad \text{as} \quad z \to -\infty. \end{aligned}\right\} \tag{4.38}$$

The solution to (4.38) may be obtained directly, which leads to the solution of the original eigenvalue problem (4.32), (4.33) as $u(z) = 1 - e^{z/\surd 2}$ $(-\infty < z < 0)$ with $v^* = \frac{1}{\surd 2}$. The minimum speed travelling wave for the modified Fisher equation therefore has

$$v^* = \tfrac{1}{\surd 2}, \tag{4.39 a}$$

with

$$u(z) = \begin{cases} 1 - e^{z/\surd 2}, & -\infty < z < 0, \\ 0, & z \geqslant 0. \end{cases} \tag{4.39 b}$$

The faster travelling waves, with $v > \frac{1}{\surd 2}$ are classical (via theorem (4.27)) and we have been unable to construct analytical forms for these. However, they are readily computed via numerical integration (with an iterative shooting method) of a two point boundary value problem, and examples are given in figure 4 with $v = 0.8$, 1.0, 2.0.

We now return to discussion of the full initial boundary value problem IBVP.

5. Qualitative behaviour of the solution to IBVP

We consider first the minimum speed travelling wave solution of the integral conservation law (2.1) discussed in §4. We denote this by

$$u \equiv T(z) = \begin{cases} H(z), & z < 0, \\ 0, & z \geqslant 0, \end{cases} \tag{5.1}$$

where $z = x - v^*t$, $H(0) = 0$, $H'(0) = -v^*$, $H(z)$ is monotonically decreasing in $z < 0$ and

$$H(z) \sim 1 - H_{-\infty}\, \mathrm{e}^{\lambda_2 z} \quad \text{as} \quad z \to -\infty \tag{5.2}$$

for some fixed constant $H_{-\infty} > 0$. To obtain some insight into the qualitative behaviour of the solution to IBVP, we consider a closely related initial value problem, which we shall denote by IVP. The problem IVP is obtained from IBVP by extending the domain to $-\infty < x \leqslant s(t)$, $t > 0$ and replacing boundary condition (2.18) with

$$u(x,t) \to 1 \quad \text{as} \quad x \to -\infty, \quad t > 0, \tag{5.3}$$

which allows the reaction to have reached completion as $x \to -\infty$. A modification is also required to the initial condition (2.17*a*). We extend $g(x)$ continuously into $x < 0$, so that

$$T(x) \leqslant u_0\, g(x) < 1, \quad x < 0, \tag{5.4a}$$

$$u_0\, g(x) \sim 1 - g_{-\infty}\, \mathrm{e}^{\lambda_2 x} \quad \text{as} \quad x \to -\infty, \tag{5.4b}$$

with $g_{-\infty} < H_{-\infty}$. We can now apply the comparison theorem of Oleinik *et al.* (1958) (which extends the usual comparison theorem for regular parabolic operators to weak solutions of singular parabolic operators of the type discussed here) to IVP. We readily observe that

$$\underline{u}(x,t) = T(x - v^*t) \tag{5.5}$$

provides a lower solution to IVP, whereas

$$\bar{u}(x,t) = T(x - v^*t - x_0) \tag{5.6}$$

provides an upper solution to IVP, with x_0 chosen sufficiently large so that $u_0\, g(x) \leqslant T(x - x_0)$ for all $-\infty \leqslant x \leqslant \sigma$. (Note that this certainly requires that $x_0 > \max\{\sigma, \lambda_2^{-1} \log(H_{-\infty}/g_{-\infty})\}$.) We thus have

Theorem 5.7. *Let $u(x,t)$, $s(t)$ be a solution of IVP then*

$$T(x - v^*t) \leqslant u(x,t) \leqslant T(x - v^*t - x_0) \quad \textit{for} \quad -\infty < x < \infty, \quad t \geqslant 0.$$

Moreover,

$$u(\xi + ct, t) \to \begin{cases} 0 & \textit{for finite} \quad t, \quad c > v^*, \\ 1 & \textit{as} \quad t \to \infty, \quad c < v^*, \end{cases} \tag{5.8}$$

for any $-\infty < \xi < \infty$ fixed, and

$$\max\{\sigma, v^*t\} \leqslant s(t) \leqslant v^*t + x_0, \tag{5.9}$$

for all $t \geqslant 0$.

Proof. Follows from Oleinik *et al.* (1958) with (5.5, 6). (5.8) and (5.9) follow from properties of $T(z)$ given in §4. □

From theorem (5.7) we observe that

$$s(t) = v^*t + O(1) \quad \text{as} \quad t \to \infty, \tag{5.10}$$

with the minimum speed travelling wave structure developing in IVP as $t \to \infty$.

We can now relate theorem (5.7) to IBVP. We note that $T_x(-v^*t - x_0) < 0$ for all $t > 0$, and so $T(x - v^*t - x_0)$ also provides an upper solution to IBVP in $x, t \geqslant 0$. In addition, $T(-v^*t) - u(0,t) < 0$ in $t > 0$, provided u_0 is sufficiently close to unity. This ensures that $T(x - v^*t)$ provides a lower solution to IBVP in x, $t \geqslant 0$. Therefore theorem (5.7) continues to hold for IBVP, provided that u_0 is taken sufficiently close to unity.

We next consider the details of the solution to IBVP as $t \to 0$.

6. Asymptotic solution to IBVP as $t \to 0$

In this section we consider a formal asymptotic solution to (2.16)–(2.19) as $t \to 0$. The behaviour depends critically on the nature of $g(x)$ as $x \to \sigma^-$, (2.7), and we find there are three distinct cases to consider.

We begin by considering the case when $m = 1$ in (2.7), so that

$$g(x) \sim u_0^{-1} g_1 (x - \sigma) \quad \text{as} \quad x \to \sigma^-, \tag{6.1}$$

for some $g_1 < 0$. (2.16) then suggests looking for an expansion in the form

$$u(x,t) \sim \sum_{n=0}^{\infty} u_n(x)\, t^n, \quad 0 < x < \sigma - o(1) \tag{6.2}$$

as $t \to 0$, with

$$s(t) \sim \sigma + \sum_{m=1}^{\infty} s_m \phi_m(t) \tag{6.3}$$

as $t \to 0$. Here $\phi_{p+1}(t) = o(\phi_p(t))$ as $t \to 0$. On substituting from (6.2) into (2.16) we obtain $u_0(x) \equiv u_0 g(x)$, $u_1(x) = u_0 D(u_0 g(x))\, g''(x) + u_0^2 D'(u_0 g(x))\, [g'(x)]^2 + R(u_0 g(x))$, so that from (6.2),

$$u(x,t) = u_0 g(x) + t\{u_0 D(u_0 g(x))\, g''(x) + u_0^2 D'(u_0 g(x))\, [g'(x)]^2 + R(u_0 g(x))\} + O(t^2) \tag{6.4}$$

as $t \to 0$ with $0 < x < \sigma - o(1)$. This expansion remains uniform when $0 < x < \sigma - o(1)$; however, on using (6.1), we observe a non-uniformity as $x \to \sigma^-$. In particular, this non-uniformity occurs when $x = \sigma + O(t)$ and $u = O(t)$ as $t \to 0$. Therefore, to complete the asymptotic structure as $t \to 0$, we must include an edge region. In the edge region we introduce the scaled variables $\bar{u}$ and $\bar{x}$, where

$$x = \sigma + t\bar{x}, \quad u = t\bar{u}(\bar{x}, t), \tag{6.5}$$

with $\bar{x}, \bar{u} = O(1)$ as $t \to 0$. In the edge region, we look for an expansion in the form

$$\bar{u}(\bar{x}, t) = \bar{u}_0(\bar{x}) + o(1), \quad -\infty < \bar{x} < s_1, \tag{6.6}$$

as $t \to 0$, after noting from (6.5) and (6.2) that

$$\phi_1(t) \equiv t. \tag{6.7}$$

On substituting from (6.5), (6.6), (6.7), (6.3) into (2.16), (2.18), the leading order problem in the edge region becomes

$$\bar{u}_0\,\bar{u}_{0_{\bar{x}\bar{x}}}+(\bar{u}_{0_{\bar{x}}})^2+\bar{x}\bar{u}_{0_{\bar{x}}}-\bar{u}_0=0,\quad -\infty<\bar{x}<s_1, \tag{6.8a}$$

$$\bar{u}_0(s_1)=0,\quad \bar{u}_{0_{\bar{x}}}(s_1)=-s_1,\quad \bar{u}_0(\bar{x})\sim g_1\,\bar{x}\quad\text{as}\quad \bar{x}\to-\infty. \tag{6.8b–d}$$

Here condition (6.8*d*) arises from matching to expansion (6.2) as $\bar{x}\to-\infty$. It is readily verified that an exact solution to the eigenvalue problem (s_1 is the eigenvalue) (6.8) is given by

$$\bar{u}_0(\bar{x})=g_1(\bar{x}+g_1),\quad s_1=-g_1. \tag{6.9}$$

Thus, in the edge region we have

$$u(\bar{x},t)=g_1(\bar{x}+g_1)\,t+O(t^2),\quad -\infty<\bar{x}<-g_1, \tag{6.10a}$$

with

$$s(t)=\sigma-g_1\,t+O(t^2), \tag{6.10b}$$

as $t\to 0$.

To summarize, we observe that the development of $u(x,t)$ in the support domain $0\leqslant x\leqslant s(t)$ has a double structure as $t\to 0$. For $0\leqslant x\leqslant \sigma-O(t)$, $u(x,t)$ has a regular expansion with an $O(t)$ correction to its initial form $u_0\,g(x)$, (6.4). However, in the edge region where $x=\sigma+O(t)$ then (6.10*a*) and (6.10*b*) show that the boundary of the support domain at $x=s(t)$ initiates propagation at $t=0^+$ with an impulsive velocity $\dot{s}(t)\sim -g_1+O(t)$ as $t\to 0^+$. In this case, the support domain initiates expansion immediately ($t=0^+$) with finite speed $-g_1$.

We next consider the case when $m=0$ in (2.7) so that

$$g(x)\sim u_0^{-1}g_0+O([x-\sigma]),\quad\text{as}\quad x\to\sigma^-, \tag{6.11}$$

for some $g_0>0$. Again, away from the edge region, we look for an asymptotic expansion for $u(x,t)$ in the form of (6.2), with an expansion for $s(t)$ following (6.3). The expansion for $u(x,t)$ is then given by (6.4). An examination of (6.4), using (6.11), again reveals a non-uniformity as $x\to\sigma^-$. However, in this case the non-uniformity occurs when $x=\sigma+O(t^{\frac{1}{2}})$ and $u=O(1)$ as $t\to 0$. Therefore we introduce an edge region with the scaled variable $\bar{x}$ defined by

$$x=\sigma+t^{\frac{1}{2}}\bar{x}, \tag{6.12}$$

and $\bar{x}=O(1)$ as $t\to 0$ in this region. We expand u in the following form:

$$u(\bar{x},t)=\hat{u}_0(\bar{x})+o(1),\quad -\infty<\bar{x}<s_1, \tag{6.13}$$

as $t\to 0$, after noting from (6.12) and (6.3) that in this case,

$$\phi_1(t)\equiv t^{\frac{1}{2}}. \tag{6.14}$$

On substitution from (6.12), (6.13), (6.14), (6.3) into (2.16), (2.19*a*, *b*), we obtain the leading order problem in the edge region as

$$\begin{gathered}[D(\hat{u}_0)\,\hat{u}_{0_{\bar{x}}}]_{\bar{x}}+\tfrac{1}{2}\bar{x}\hat{u}_{0_{\bar{x}}}=0,\quad -\infty<\bar{x}<s_1,\\ \hat{u}_0(s_1)=0,\quad \hat{u}_{0_{\bar{x}}}(s_1)=-\tfrac{1}{2}s_1,\quad \hat{u}_0(\bar{x})\to g_0\quad\text{as}\quad \bar{x}\to-\infty.\end{gathered} \tag{6.15}$$

We have been unable to make further direct progress with the eigenvalue problem (6.15). However, we have investigated (6.15) in more detail for the specific case when $D(u)\equiv u$. In this case the solution to (6.15) may be written in the form

$$\hat{u}_0(\bar{x})=g_0\,H(g_0^{-\frac{1}{2}}\bar{x}),\quad -\infty<\bar{x}<s_1,$$

$$s_1=g_0^{\frac{1}{2}}\lambda,$$

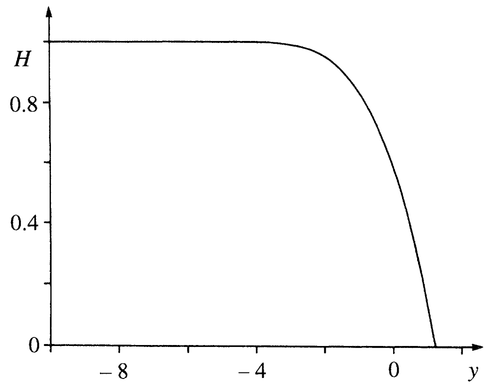

Figure 5. A graph of $H(\eta)$ against η with $-\infty<\eta<\lambda=1.2385$.

where $H(\eta)$ and λ satisfy the eigenvalue problem

$$\left.\begin{array}{l} HH''+(H')^2+\tfrac{1}{2}\eta H'=0, \quad -\infty<\eta<\lambda, \\ H(\lambda)=0, \quad H'(\lambda)=-\tfrac{1}{2}\lambda, \quad H(\eta)\to 1 \quad \text{as} \quad \eta\to-\infty, \end{array}\right\} \tag{6.16}$$

which now has no parameters except the eigenvalue λ (with prime $\equiv \mathrm{d}/\mathrm{d}\eta$). A numerical solution of (6.16) has been obtained using a shooting method and a graph of $H(\eta)$ is shown in figure 5; the eigenvalue λ was determined as

$$\lambda \approx 1.2385. \tag{6.17}$$

Hence, in this case, for general $D(u)$, we have from (6.3) and (6.14) that

$$s(t)=\sigma+s_1 t^{\frac{1}{2}}+O(t) \quad \text{as} \quad t\to 0, \tag{6.18}$$

with $s_1=1.2385 g_0^{\frac{1}{2}}$ when $D(u)\equiv u$. Thus the boundary of the support domain, at $x=s(t)$, again initiates propagation at $t=0^+$, now with an unbounded velocity $\dot{s}(t)\sim \frac{1}{2}s_1 t^{-\frac{1}{2}}+O(1)$ as $t\to 0^+$. At the edge of the support domain we have,

$$u_x(s(t),t)\sim -\tfrac{1}{2}s_1 t^{-\frac{1}{2}} \tag{6.19}$$

as $t\to 0^+$, which for the case $D(u)\equiv u$ reduces to

$$u_x(s(t),t)\sim -0.6193(g_0/t)^{\frac{1}{2}}, \tag{6.20}$$

as $t\to 0^+$.

The next case is when $m=2$ in (2.7), so that

$$g(x)\sim u_0^{-1} g_2 (x-\sigma)^2+O([x-\sigma]^3), \quad \text{as} \quad x\to\sigma^-, \tag{6.21}$$

for some $g_2>0$. Again, away from the edge region, we look for an asymptotic expansion for $u(x,t)$ in the form of (6.2). After substitution from (6.2) into (2.16) we arrive at (6.4) for $u(x,t)$, to $O(t)$ as $t\to 0$. However, in this case as $x\to\sigma^-$,

$$u_0(x)\sim g_2(x-\sigma)^2, \quad u_1(x)\sim g_2(1+6g_2)(x-\sigma)^2, \tag{6.22}$$

with further calculations showing that

$$u_i(x)\sim O[(x-\sigma)^2] \quad \text{as} \quad x\to\sigma^-, \tag{6.23}$$

for $i=2,3,\ldots$. Therefore, in contrast with the two previous cases, the regular expansion for $u(x,t)$, (6.2), remains uniform as $x\to\sigma^-$, and an edge region as $t\to 0$ is not present in this case. Moreover, from (6.22), (6.23) and (6.2) we have that the edge of the support domain is at $x = \sigma$, and there $u_i(\sigma) = 0$, $i = 0, 1, \ldots$. Thus, in (6.3) $\phi_m(t) = t^m$, $m = 1, 2, 3, \ldots$ after which conditions (2.19a, b) require $s_m = 0$, $m = 1, 2, \ldots$.

To summarize, in this case $s(t) \equiv \sigma$ for small t, and so the support domain does not expand initially ($t = 0^+$). The only development in the small time in $u(x,t)$ is confined to the initial support domain $0 \leqslant x \leqslant \sigma$ and is given through the expansion (6.4) which remains uniform as $t \to 0$ throughout the whole support domain. We may conclude that if the support domain is extended in this case, the initiation of this expansion must occur at finite $t (> 0)$. We return to this question in the next section.

Finally, we have the cases with $m = 3, 4, \ldots$ in (2.7), so that

$$g(x) \sim u_0^{-1} g_n (x - \sigma)^n + O([x - \sigma]^{n+1}), \quad \text{as} \quad x \to \sigma^-, \tag{6.24}$$

for some g_n (positive for n even and negative for n odd) and $n = 3, 4, \ldots$. As in the previous cases we expand $u(x,t)$ in the form of (6.2) and arrive at the development (6.4), up to $O(t)$ as $t \to 0$. As in the case with $m = 2$, we find that as $x \to \sigma^-$

$$u_0(x) \sim g_n (x - \sigma)^n, \quad u_1(x) \sim g_n (x - \sigma)^n, \tag{6.25}$$

with higher order terms having the form

$$u_i(x) \sim O([x - \sigma]^n), \tag{6.26}$$

for $i = 2, 3, 4, \ldots$. Thus in this case (as for $m = 2$) the expansion (6.2) remains uniform as $x \to \sigma^-$ and an edge region as $t \to 0$ is not present. An immediate consequence of this is that $\phi_j(t) \equiv t^j$ and $s_j = 0$ for $j = 1, 2, \ldots$ in (6.3). Hence $s(t) \equiv \sigma$ for small t and the support domain does not expand initially ($t = 0^+$), the detailed development following that described for the previous case $m = 2$.

In this section we have uncovered two distinct types of behaviour in $u(x,t)$ for small t. For initial data $g(x)$ with $m = 0$ or 1, then the support of $u(x,t)$ expands immediately, according to (6.10b) or (6.18) respectively. However, for initial data with $m \geqslant 2$, the support of $u(x,t)$ remains stationary when $0 < t \ll 1$.

The solution thus exhibits a 'waiting time' for initial data with $m \geqslant 2$. This phenomenon has been studied extensively in the case of nonlinear diffusion, when $D(u) = u^n$, $R(u) \equiv 0$, $n = 1, 2, \ldots$; in particular see Lacey *et al.* (1982), Lacey (1983), Aronson *et al.* (1983), Kath & Cohen (1982) and Lacey & Vazquez (1992). We examine the waiting time for the present problem in the next section.

As a final remark we observe that the expansion (6.4), valid for all cases $m \geqslant 0$ as $t \to 0$ with $0 < x < \sigma - o(1)$ has a further non-uniformity as $x \to 0^+$, over which the zero flux boundary condition (2.18) is satisfied. A further passive region can be introduced to accommodate this non-uniformity. The details are not given here as they follow directly those given in King & Needham (1992) for a similar problem.

7. Waiting times in the cases $m \geqslant 2$

In §6 we have formally established the existence of a waiting time ($s(t) \equiv \sigma$ for $0 \leqslant t \leqslant t_\omega$, with $t_\omega > 0$) in the cases $m \geqslant 2$. To analyse this further, we first restrict attention to the case $D(u) \equiv u$, and limit ourselves to the modified problem IVP, allowing us to make use of the comparison theorem of Oleinik *et al.* (1958).

To begin we observe that

$$V_\lambda(x,t) = \begin{cases} e^t[(\lambda^{-1} + 6) - 6\,e^t]^{-1}(x - \sigma)^2, & -\infty < x \leqslant \sigma, \\ 0, \quad x > \sigma \end{cases} \tag{7.1}$$

is a classical solution of

$$V_t = (VV_x)_x + V, \tag{7.2}$$

on $-\infty < x < \infty$, $0 \leqslant t < t_\lambda$, with,

$$t_\lambda = \log(1 + \tfrac{1}{6}\lambda^{-1}). \tag{7.3}$$

Moreover,

$$V_\lambda(x,0) = \begin{cases} \lambda(x-\sigma)^2, & -\infty < x \leqslant \sigma, \\ 0, & x > \sigma, \end{cases} \tag{7.4}$$

and so in the case $m \geqslant 2$ we may choose $\lambda = \lambda_0$ sufficiently large so that $u_0 g(x) \leqslant \lambda_0(x-\sigma)^2$ on $-\infty < x \leqslant \sigma$. In fact we choose

$$\lambda_0 = \sup_{x \in (-\infty, \sigma)} [u_0 g(x)\,(x-\sigma)^{-2}],$$

noting that λ_0 is bounded above for the cases $m \geqslant 2$, and $\lambda_0 \geqslant g_2$ when $m = 2$. With condition (2.3*b*) on $R(u)$ it is then readily established that $V_{\lambda_0}(x,t)$ is an upper solution to IVP for $0 \leqslant t < t_{\lambda_0}$. Also, as demonstrated in §5, $T(x - V^*t)$ provides a lower solution to IVP. Thus, with $u(x,t)$ being a solution to IVP in the cases $m \geqslant 2$, we have

$$T(x - V^*t) \leqslant u(x,t) \leqslant V_{\lambda_0}(x,t), \tag{7.5}$$

on $-\infty < x < \infty$, $0 \leqslant t < t_{\lambda_0}$, via the comparison theorem of Oleinik *et al.* (1958) (note that (7.5) implies that $\lambda_0 \geqslant \frac{1}{6}(e^{\sigma/V^*} - 1)$). In particular we have

Theorem 7.6. *Let $u(x,t)$ be a solution of IVP (with $D(u) \equiv u$) in the cases $m \geqslant 2$. Then $u(x,t)$ has a waiting time t_ω with*

$$\log(1 + \tfrac{1}{6}\lambda_0^{-1}) \leqslant t_\omega \leqslant V^{*-1}\sigma. \tag{7.7}$$

That is $s(t) \equiv \sigma$ for $0 \leqslant t \leqslant t\omega$.

Proof. Follows directly from (7.5) and (3.6). □

As in §5 it is readily established that $V_{\lambda_0}(x,t)$ provides an upper solution to IBVP, as $V_{\lambda_0 x}(0,t) < 0$ for all $0 < t < t_{\lambda_0}$, whereas $T(x - V^*t)$ provides a lower solution for u_0 sufficiently close to unity. Thus, theorem (7.6) holds also for IBVP, when the initial data has u_0 sufficiently close to unity. In fact a lower solution can be constructed for IBVP for any u_0, in terms of one of the similarity solutions given in Lacey *et al.* (1982). We observe that

$$\omega(x,t) = \begin{cases} \dfrac{\sigma^2}{4C^{\frac{2}{3}}(6t+C)^{\frac{1}{3}}} - \dfrac{x^2}{(6t+C)}, & 0 \leqslant x \leqslant \dfrac{\sigma(6t+C)^{\frac{1}{3}}}{2C^{\frac{1}{3}}}, \\ 0, & x > \sigma(6t+C)^{\frac{1}{3}}/2C^{\frac{1}{3}}, \end{cases} \tag{7.8}$$

is a piecewise classical solution to the parabolic equation

$$\omega_t = (\omega\omega_x)_x, \quad x, t > 0, \tag{7.9}$$

for any $C > 0$. Now for $C = C_0$ sufficiently large (certainly $C_0 \geqslant \sigma^2/4u_0$),

$$u_0 g(x) \geqslant (\tfrac{1}{4}\sigma^2 - x^2)/C_0$$

on $0 \leqslant x \leqslant \sigma$. Moreover, $\omega_x(0,t) = 0$ on $t \geqslant 0$. Hence, as $\omega(x,t) \leqslant 1$ on $x, t \geqslant 0$, then $R(\omega) \geqslant 0$ on $x, t \geqslant 0$, and $\omega(x,t)$ forms a sub-solution to IBVP on $x, t \geqslant 0$ (Oleinik *et al.* 1958). Therefore, a modified version of theorem (7.6) for IBVP is

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Theorem 7.10. *Let $u(x,t)$ be a solution of* IBVP (*with $D(u) \equiv u$*) *in the cases $m \geqslant 2$. Then $u(x,t)$ has a waiting time t_ω with*

$$\log(1+\tfrac{1}{6}\lambda_0^{-1}) \leqslant t_\omega \leqslant \tfrac{7}{6}C_0. \tag{7.11}$$

We now examine the cases $m = 2$, $m \geqslant 3$ separately, for IBVP.

$m = 2$

Here $u_0 g(x) \sim g_2(x-\sigma)^2$ as $x \to \sigma^-$ with $g_2 > 0$, and from (7.11) $t_\omega \geqslant \log(1+\tfrac{1}{6}\lambda_0^{-1})$ with $\lambda_0 = \sup_{0 \leqslant x \leqslant \sigma}\{u_0 g(x)(x-\sigma)^{-2}\}$. In this case $\lambda_0 \geqslant g_2$, and for data with $\lambda_0 = g_2$ we obtain the optimum lower bound on t_ω as

$$t_\omega \geqslant \log[1+\tfrac{1}{6}g_2^{-1}]. \tag{7.12}$$

Thus, for suitable initial data, the waiting time can be made arbitrarily large (but finite via (7.11)). However, it should be noted that in this case, initial data may also be chosen so that t_ω is arbitrarily small (but non-zero).

$m \geqslant 3$

Here $u_0 g(x) \sim g(x-\sigma)^m$ as $x \to \sigma^-$ with $(-1)^m g_m > 0$. In this case we have $t_\omega \geqslant \log[1+\tfrac{1}{6}\lambda_0^{-1}]$, and again for suitable initial data, the waiting time can be made arbitrarily large or small (but finite and non-zero).

We next consider in detail the behaviour of the solution to IBVP at the edge of the support domain in the cases $m \geqslant 2$.

8. Local solution to IBVP as $x \to \sigma^-$ with $0 \leqslant t < t_\omega$ in the cases $m \geqslant 2$

We examine the development of the edge of the support domain in the cases $m \geqslant 2$ by attempting to construct a local solution to IBVP as $x \to \sigma^-$ with $0 \leqslant t < t_\omega$. We have

$$s(t) \equiv \sigma, \quad 0 \leqslant t \leqslant t_\omega. \tag{8.1}$$

With the initial data analytic at $x = \sigma$, we expect that $u(x,t)$ will remain analytic at $x = \sigma$ until the waiting time is reached, that is on $0 \leqslant t < t_\omega$. Under these conditions, $u(x,t)$ will have a convergent power series expansion in x about $x = \sigma$ for each $0 \leqslant t < t_\omega$, with radius of convergence $R(t) > 0$ on $0 \leqslant t < t_\omega$, and $R(t) \to 0$ as $t \to t_\omega^-$. Moreover, the coefficients of this power series expansion will be analytic functions of $t \in [0, t_\omega)$. We consider first,

(*a*) $m = 2$

Following the above discussion, we look for a power series expansion of $u(x,t)$ about $x = \sigma$ in the form,

$$u(x,t) = \sum_{n=2}^{\infty} (x-\sigma)^n \chi_n(t), \tag{8.2}$$

with $0 \leqslant \sigma - x < R(t)$, $0 \leqslant t < t_\omega$.

We observe immediately that (8.1) and (8.2) satisfy the support edge boundary conditions of IBVP, (2.19*a*, *b*). We now substitute from (8.1) into equation (2.16) and initial condition (2.17*a*, *b*) with (2.7). At leading order we obtain the initial-value problem

$$\chi_2' = 6\chi_2^2 + \chi_2, \quad \chi_2(0) = g_2, \quad t \geqslant 0. \tag{8.3}$$

The solution to (8.3) is readily obtained as

$$\chi_2(t) = e^t/[(g_2^{-1}+6) - 6e^t]. \tag{8.4}$$

On proceeding to higher order we find that

$$\chi_3(t) = g_3\,\mathrm{e}^t/[g_2\{(g_2^{-1}+6)-6\,\mathrm{e}^t\}]^{\frac{10}{3}}, \tag{8.5a}$$

$$\chi_4(t) = \frac{15g_3^2\,\mathrm{e}^t}{4g_2^{\frac{20}{3}}(\gamma-6\,\mathrm{e}^t)^{\frac{17}{3}}}+\frac{(g_4(\gamma-6)^5+\delta)\,\mathrm{e}^t}{(\gamma-6\,\mathrm{e}^t)^5}-\frac{5D''(0)\,\mathrm{e}^{2t}}{18(\gamma-6\,\mathrm{e}^t)^2}-\frac{5D''(0)\,\mathrm{e}^t}{432(\gamma-6\,\mathrm{e}^t)}-\frac{R''(0)\,\mathrm{e}^t}{48(\gamma-6\,\mathrm{e}^t)}, \tag{8.5b}$$

where δ is a constant given by

$$\delta = \tfrac{5}{18}D''(0)\,(\gamma-6)^3+\tfrac{5}{432}D''(0)\,(\gamma-6)^4-\frac{15}{4}\frac{g_3^2}{g_2^{\frac{20}{3}}(\gamma-6)^{\frac{2}{3}}}+\tfrac{1}{48}R''(0)\,(\gamma-6)^4, \tag{8.6}$$

and

$$\gamma = g_2^{-1}+6. \tag{8.7}$$

It is now readily observed that all higher order coefficients $\chi_n(t)$, $n = 5, 6, \ldots$, have the divisor $[\gamma-6\,\mathrm{e}^t]$, with, as $t\rightarrow\log(\gamma/6)$,

$$\chi_n(t) = O[(\log(\gamma/6)-t)^{-(7n-11)/3}]. \tag{8.8}$$

On putting $t_c = \log(\gamma/6)$, we write

$$\chi_n(t) = [(t_c-t)]^{-(7n-11)/3}\,a_n(t), \quad n = 2, 3, \ldots, \tag{8.9}$$

where $a_n(t)$ is now a bounded function on $t\in[0,T]$ for any $T>0$. We have

$$0 \leqslant R(t) \leqslant (t_c-t)^{\frac{7}{3}}\,a(t), \tag{8.10}$$

where

$$a(t) = \limsup_{n\rightarrow\infty}\left|\frac{a_n(t)}{a_{n+1}(t)}\right|. \tag{8.11}$$

The inequality (8.10) indicates that in this case $t_\omega \leqslant t_c$, with equality holding when $a(t)$ is bounded above zero for $t\in[0,t_c]$. In particular we have from §7 that when

$$\sup_{0\leqslant x\leqslant\sigma}\{u_0\,g(x)\,(x-\sigma)^{-2}\} = g_2,$$

then

$$t_\omega = t_c = \log[1+\tfrac{1}{6}g_2^{-1}], \tag{8.12}$$

via (7.12).

It should also be noted that the development (8.2) provides a formal asymptotic solution to equation (2.16) as $x\rightarrow\sigma^-$ with $0\leqslant t<t_c$. In the case when $t_\omega<t_c$, we expect that this expansion cannot be continued into $u(x,t)$ when $x=\sigma-O(1)$ for $t>t_\omega$, and (although it still remains as a formal asymptotic solution to equation (2.16) as $x\rightarrow\sigma^-$, $0\leqslant t<t_c$) then fails to be an asymptotic representation of $u(x,t)$, the solution of IBVP in $t>t_\omega$; that is as $t\sim t_\omega-o(1)$, the global structure of $u(x,t)$ from the interior of the support domain, $x=\sigma-O(1)$, interacts with the local behaviour at the edge of the support domain, $x=\sigma-o(1)$. An asymptotic framework for this type of behaviour has been given by Lacey (1983) for a class of nonlinear diffusion equations and we expect this structure to be preserved in the present reaction–diffusion context. In particular, this analysis shows that as $t\rightarrow t_\omega^+$,

$$\dot{s}(t)\rightarrow C_\omega,$$

for some $C_\omega>0$, which depends upon $g(x)$, and the edge of the support domain initiates its motion at $t=t_\omega$ with a finite speed.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

In the case when (8.12) holds, then (8.2) can be regarded as an asymptotic expansion of $u(x,t)$ as $x \to \sigma^-$ that remains valid until $t \sim t_c - o(1)$, when a non-uniformity develops due to the singularities in $\chi_n(t)$ as $t \to t_c^-$ (8.8). We can continue the development of $u(x,t)$ as $x \to \sigma^-$ with $t = t_c - o(1)$ by introducing a further asymptotic region. Expansion (8.1) has the form

$$u(x,t) \sim \frac{(x-\sigma)^2}{6(t_\omega - t)} + \frac{g_3(x-\sigma)^3}{6g_2^{\frac{10}{3}}\gamma^{\frac{7}{3}}(t_\omega - t)^{\frac{10}{3}}} + \left\{\frac{15g_3^2}{24_2^{\frac{20}{3}}\gamma^{\frac{14}{3}}(t_\omega - t)^{\frac{17}{3}}} + \frac{(g_4(\gamma-6)+\delta)^5}{6\gamma^4(t_\omega - t)^5}\right.$$
$$\left. - \frac{5D''(0)}{648(t_\omega - t)^2} - \left[\frac{5D''(0)}{2592} + \frac{R''(0)}{2304}\right]\frac{1}{(t_\omega - t)}\right\} \times (x-\sigma)^4 + \ldots, \quad (8.13)$$

as $x \to \sigma^-$ with $0 < (t_\omega - t) \ll 1$. The form of (8.13) indicates that there are two cases to be considered.

(i) $g_3 \neq 0$

With $g_3 \neq 0$, (8.13) indicates that expansion (8.2) becomes non-uniform when

$$x = \sigma + O[(t_\omega - t)^{\frac{7}{3}}], \quad (8.14a)$$

with

$$u = O[(t_\omega - t)^{\frac{11}{3}}]. \quad (8.14b)$$

(ii) $g_3 = 0$

Expansion (8.2) now becomes non-uniform when

$$x = \sigma + O[(t_\omega - t)^2], \quad (8.15)$$

with

$$u = O[(t_\omega - t)^3]. \quad (8.16)$$

We need to introduce two asymptotic regions when $x \to \sigma^-$ and $t \to t_\omega$. The first of these we will call the following.

Edge transition region I

In this region we have $0 \leqslant (\sigma - x) \ll 1$ and $0 < t_\omega - t \ll 1$. In particular we put

$$x = \sigma + (t_\omega - t)^\alpha \eta, \quad \eta \leqslant 0, \quad (8.17)$$

and expand

$$u(\eta, t) = (t_\omega - t)^{2\alpha-1}\tilde{u}(\eta) + o[(t_\omega - t)^{2\alpha-1}], \quad (8.18)$$

as $t \to t_\omega^-$ with $\eta = O(1)$. The forms (8.17), (8.18) are motivated by (8.14)–(8.16) with

$$\left.\begin{array}{lll} \alpha = \frac{7}{3} & \text{for} & \text{case (i)}, \\ \alpha = 2 & \text{for} & \text{case (ii)}. \end{array}\right\} \quad (8.19)$$

On substituting from (8.19), (8.18) into equation (2.16) we obtain at leading order as $t \to t_\omega^-$,

$$\tilde{u}\tilde{u}_{\eta\eta} + (\tilde{u}_\eta)^2 - \alpha\eta\tilde{u}\eta + (2\alpha - 1)\,\tilde{u} = 0, \quad \eta < 0. \quad (8.20)$$

Now, (8.18) must satisfy the support edge boundary conditions of IBVP, (2.19a, b), together with matching to (8.13) as $\eta \to 0^-$ and matching to the solution in the bulk of the support domain ($t \sim t_\omega$, $x \sim \sigma - O(1)$) as $\eta \to -\infty$. The matching conditions require that,

$$\tilde{u}(\eta) \sim \eta^2/6 \quad \text{as} \quad \eta \to 0^-, \quad (8.21)$$

$$\tilde{u}(\eta) \sim C[-\eta]^{2-(1/\alpha)} \quad \text{as} \quad \eta \to -\infty, \quad (8.22)$$

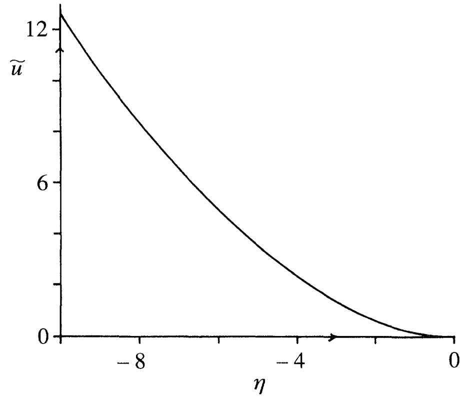

Figure 6. The numerical solution of (8.16)–(8.18) with $C = 1$ and $\alpha = 2$.

with the latter condition giving $u \sim O(1)$ when $x \sim \sigma - O(1)$ and $t \sim t_\omega$. The support edge conditions, (2.19a, b), together with (8.21) then require that,

$$s(t) \equiv 0, \quad 0 < t_\omega - t \ll 1. \tag{8.23}$$

The constant C is undetermined in this asymptotic expansion, and represents the influence of the solution when $x \sim \sigma - O(1)$ and $t \sim O(1)$ on the solution in the support edge region. A numerical solution of (8.20)–(8.22) (using a Runge–Kutta method starting from the boundary condition at $-\infty$) with $C = 1$ and $\alpha = 2$ is shown in figure 6.

The boundary condition (8.22) can be developed to higher order through (8.20), which gives

$$\tilde{u}(\eta) \sim C(-\eta)^{2-(1/\alpha)} - C^2(2-\alpha^{-1})(3-2\alpha^{-1})(-\eta)^{2-(2/\alpha)} + \ldots, \tag{8.24}$$

as $\eta \to -\infty$. We now use (8.24) to examine the behaviour of u as $t \to t_\omega^-$ with $0 < (\sigma - x) \ll 1$. From (8.18) and (8.24) we arrive at

$$u(x,t) \sim C(\sigma-x)^{2-(1/\alpha)} - C^2(2-\alpha^{-1})(3-2\alpha^{-1})(\sigma-x)^{2-(2/\alpha)}(t_\omega - t) + O[(t_\omega - t)^2] \tag{8.25}$$

as $t \to t_\omega^-$ with $0 < (\sigma - x) \ll 1$ fixed. Note that (8.25) develops as a regular power series in $(t_\omega - t)$. To continue the solution into $0 < (t - t_\omega) \ll 1$ with $|x - \sigma| \ll 1$ we must introduce the following region.

Edge transition region II

The problem in this region is to continue the solution in Edge transition region I into $0 < (t - t_\omega) \ll 1$ when $|x - \sigma| \ll 1$. Thus we must solve equation (2.16) subject to continuing with (8.25) as $t \to t_\omega^+$, which gives

$$u(x, t_\omega) \sim C(\sigma - x)^{2-(1/\alpha)}, \tag{8.26}$$

as $x \to \sigma^-$. In addition the support edge boundary conditions (2.19a,b) must be satisfied, together with matching to the solution in the bulk of the support domain ($x = \sigma - O(1)$, $t \sim t_\omega$). We note that this problem is similar to the support edge region problems for $0 < t \ll 1$ in §6 when $0 \leqslant m < 2$, with the time origin shifted to $t = t_\omega$.

To proceed we introduce the scaled variable $\tilde{\eta}$ by,

$$x = \sigma + (t - t_\omega)^\beta \tilde{\eta}, \tag{8.27}$$

with $\tilde{\eta} = O(1)$ as $t \to t_\omega^+$ and $\beta > 0$ to be determined. We expand u in the form,

$$u(\tilde{\eta}, t) = (t - t_\omega)^\nu \bar{u}(\tilde{\eta}) + \ldots, \tag{8.28}$$

as $t \to t_\omega^+$, with $\nu \geqslant 0$. To satisfy initial conditions (8.26) and balance terms in equation (2.16) we require,

$$\beta = \alpha, \quad \nu = 2\alpha - 1, \tag{8.29}$$

whilst to satisfy condition (2.19*b*) we expand $s(t)$ (the location of the edge of the support domain) as,

$$s(t) = \sigma + (t - t_\omega)^\alpha s_1 + \ldots, \tag{8.30}$$

with the constant s_1 to be determined. After substitution from (8.27), (8.28), (8.30) into (2.16), (2.19*a*, *b*), (8.26), we arrive at the leading order problem

$$\bar{u}\bar{u}_{\tilde{\eta}\tilde{\eta}} + (\bar{u}_{\tilde{\eta}})^2 - \alpha\tilde{\eta}\bar{u}_{\tilde{\eta}} - (2\alpha - 1)\,\bar{u} = 0, \quad -\infty < \tilde{\eta} < s_1, \tag{8.31}$$

$$\bar{u}(s_1) = 0, \quad \bar{u}_{\tilde{\eta}}(s_1) = -\alpha s_1, \tag{8.32a, b}$$

$$\bar{u}(\tilde{\eta}) \sim C(-\tilde{\eta})^{2-(1/\alpha)}, \quad \tilde{\eta} \to -\infty. \tag{8.33}$$

This boundary value problem has been solved numerically by an iterative shooting method in the cases $\alpha = \frac{7}{3}$, 2, and it is found that

$$s_1 = 3.8069C^{\frac{7}{3}}, \quad 2.4303C^2, \tag{8.34}$$

for $\alpha = \frac{7}{3}$, 2 respectively. Thus, the support domain initiates its expansion in this region and the initial development of the support domain depends on the cases (*i*), (*ii*) which give, via (8.30) and (8.34)

$$\dot{s}(t) \sim \begin{cases} C^{\frac{7}{3}}(t - t_\omega)^{\frac{4}{3}}, \\ C^2(t - t_\omega), \quad \text{as} \quad t \to t_\omega^+. \end{cases} \tag{8.35}$$

It remains to consider the cases with $m = 3, 4, \ldots$ in (2.7).

(*b*) $m = 3, 4, \ldots$

We again look for a power series expansion of $u(x, t)$ about $x = \sigma$ as

$$u(x, t) = \sum_{n=m}^{\infty} (x - \sigma)^n \chi_n(t), \tag{8.36}$$

with $0 \leqslant \sigma - x < \bar{R}(t)$, $0 \leqslant t < t_\omega$, and $s(t)$ given by (8.1). We observe immediately that (8.1), (8.36) satisfy the support edge boundary conditions (2.19*a*, *b*), and it remains to satisfy the initial condition (2.19*a*) with (2.7) (when $m = 3, 4, \ldots$). After substitution of (8.36) into equation (2.16) and applying the initial conditions we obtain

$$\bar{\chi}_m(t) = g_m\,\mathrm{e}^t, \quad \bar{\chi}_{m+1}(t) = (g_{m+1} - m(2m-1)\,g_m^2\,t)\,\mathrm{e}^t,$$

with, in general,

$$\left.\begin{aligned} \bar{\chi}_n(t) &= P_n(t)\,\mathrm{e}^t, \quad n = m+2, \ldots, 2m-3, \\ \bar{\chi}_n(t) &= P_n(t)\,\mathrm{e}^t + \sum_{j=2m-4}^{n-2} q_j\,\mathrm{e}^{(n-j)t}, \quad n = 2m-2, 2m-1, \ldots, \end{aligned}\right\} \tag{8.37}$$

where $P_n(t)$ is a polynomial of order $(n - m)$ and q_j is a constant.

Via Theorem 7.10, we have established that the radius of convergence of the power series (8.36), $\bar{R}(t) \to 0$ as $t \to t_\omega^-$. However, when regarded as an asymptotic expansion (8.36) continues to hold as a solution to (2.16) beyond $t = t_\omega$ (as the coefficients $\bar{\chi}_n(t)$ remain bounded with $t \in [0, T]$ for any $T > 0$), but cannot continue to represent $u(x,t)$, the solution of IBVP. It must fail to be continuable into $u(x,t)$ when $x = \sigma - O(1)$ in $t > t_\omega$. As in the case $m = 2$ with $t_\omega < t_c$, the global structure of $u(x,t)$ (on $0 \leqslant x < \sigma - O(1)$) interacts with the local expansion (8.36) as $t \to t_\omega^-$ to bring about initiation of motion at the edge of the support domain at $t = t_\omega$. Again, the structure of this interaction will follow that given by Lacey (1983), and thus has,

$$\dot{s}(t) \sim \tilde{C} \quad \text{as} \quad t \to t_\omega^+,$$

$\tilde{C} > 0$, depending upon $g(x)$. All cases have now been covered, and we confirm the asymptotic analysis by next considering a numerical solution to IBVP.

9. Numerical methods and results

To gain some further insight into the evolution of the initial data and to confirm some of the asymptotic results of the previous section it is necessary to use a numerical method to solve IBVP. The generality of the reaction and diffusion functions hitherto adopted serves no particularly useful purpose here and accordingly we consider the specific forms $D(u) \equiv u$ and $R(u) \equiv u(1-u)$, with equation (2.16) becoming

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) + u(1-u). \tag{9.1}$$

This form of reaction and diffusion functions is clearly seen to be within the set described in §2 and we further take the extent of the initial data as $\sigma = 1$. Equation (9.1) is to be solved subject to the boundary conditions

$$\partial u/\partial x\,(0,t) = 0, \quad u(s(t),t) = 0 \quad \text{and} \quad \partial u/\partial x\,(s(t),t) = -\dot{s}(t),$$

and the initial condition $u(x,0) = u_0(x)$, $0 \leqslant x \leqslant 1$. It is convenient to transform to a fixed computational domain using the transformation $\bar{x} = x/s(t)$ which gives

$$\frac{\partial u}{\partial t} - \frac{\dot{s}\bar{x}}{s}\frac{\partial u}{\partial \bar{x}} = \frac{1}{s^2}\frac{\partial}{\partial \bar{x}}\left(u\frac{\partial u}{\partial \bar{x}}\right) + u(1-u), \tag{9.2}$$

subject to the unchanged initial condition $u(\bar{x},0) = u_0(\bar{x})$ and the boundary conditions $u_{\bar{x}}(0,t) = 0$, $u(1,t) = 0$ and $u_{\bar{x}}(1,t) = -s\dot{s}$. To use existing numerical methods for classical parabolic equations the following strategy was used. The function $s(t)$ is regarded as constant over each time integration of (9.2). A local asymptotic solution to (9.1) valid near $\bar{x} = 1$ is $u = s\dot{s}(1-\bar{x}) + O((1-\bar{x})^2)$. This is used as a boundary condition at $\bar{x} = 1 - \delta$ (where δ is much smaller than any spatial discretization of (9.2)). The remaining boundary condition $u_{\bar{x}}(1,t) = -s\dot{s}$ is used only to update the value of $s(t)$ at the end of each timestep. After an integration and discretization, by the trapezium rule (the error term of which is consistent with time integration of (9.2)) the appropriate form of this condition is seen to be

$$s^2(t+\delta t) = s^2(t) - \delta t\,\{u_{\bar{x}}(1, t+\delta t) + u_{\bar{x}}(1,t)\}. \tag{9.3}$$

The advantage of using this leap-frog procedure is that the numerical solution of (9.2) can be obtained at each timestep using the NAG library routine DO2PAF which is extremely efficient and is based upon the method of lines and a Gear's procedure.

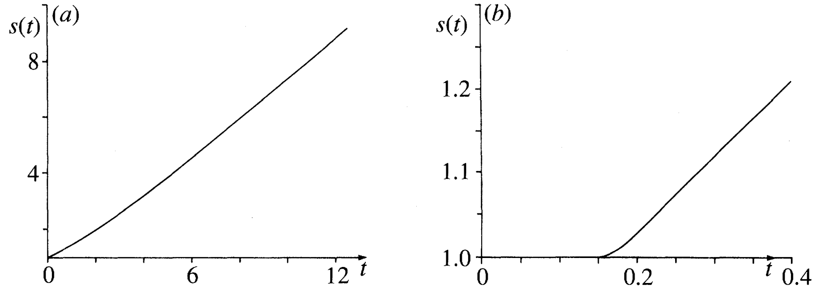

Figure 7. A graph of $s(t)$ against t for IBVP when $\sigma = 1$ and (a) $u_0 = 0.3(1-x)$; (b) $u_0 = (1-x)^2$.

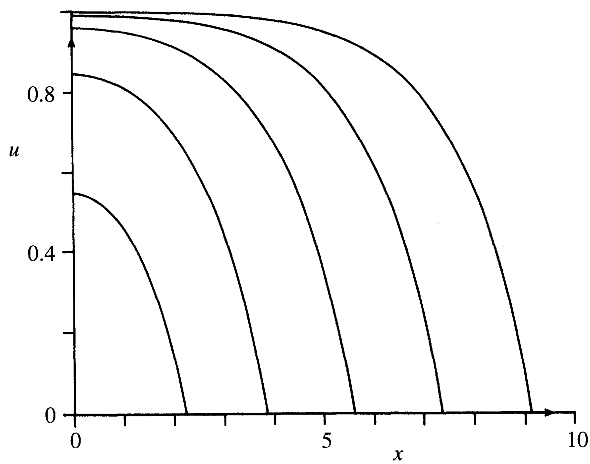

Figure 8. A graph of $u(x,t)$ against x at $t = 2.5$, 5.0, 7.5, 10.0, 12.5, when $\sigma = 1$ and $u_0(x) = 0.3\ (1-x)$.

In all the results presented here the spatial discretization of the interval $0 \leqslant \bar{x} \leqslant 1$ was taken as 0.01, the timestep was 0.005 and the quantity δ was chosen to be 10^{-6}. The choice of these particular parameters ensures that the numerical solutions are mesh independent, at least to within graphical accuracy. There are two features of the numerical solution which are of particular relevance: the behaviour of the moving boundary $s(t)$ and the long time behaviour of the solution $u(x,t)$. Figure 7*a* shows the development of $s(t)$ when the initial data was of the form of $u_0(x) = 0.31(1-x)$. It is clear that the support edge of the initial data in this case moves immediately, as is predicted by the asymptotic theory, and, in the large time limit, approaches a constant speed. This speed is computed by the above method as 0.70753 which is in good agreement with the predicted value of $1/\sqrt{2}$. For initial data of the form $u_0(x) = (1-x)^2$, figure 7*b* shows the waiting time phenomenon described in the previous section. For this case $u_0(x)$ is such that $g_2 = 1 = \sup_{0 \leqslant x \leqslant 1}\{u_0(x)\,(x-1)^{-2}\}$. Thus (8.12) holds and the theory gives $t_\omega = \log(7/6) \approx 0.154$. The computed time elapsed before $s(t)-1$ attains a value greater than 10^{-3} is 0.151 and again compares well with the theoretical value. The behaviour of $u(x,t)$ in the former case for a sequence of equally spaced times is shown in figure 8 and clearly demonstrates the approach to a non-classical travelling wave with the fully reacted state far behind the wave-front.

10. Asymptotic solution to IBVP as $t \to \infty$

In this section we examine the large time asymptotic development of IBVP when a travelling wave structure evolves as $t \to \infty$. The permanent form travelling wave which evolves from IBVP as $t \to \infty$ must have compact support, and is therefore the permanent form travelling wave of minimum speed $v = v^*$ (see §5). The asymptotic structure of $u(x,t)$ as $t \to \infty$ has three regions.

Region I

Here $x = O(1)$ as $t \to \infty$, and we expect $u(x,t) \to 1$ in this limit, which is the fully reacted state at the rear of the travelling wave. We write

$$u(x,t) = 1 + \phi(x,t), \tag{10.1}$$

where $\phi(x,t) = o(1)$ as $t \to \infty$. After substituting from (10.1) into equation (2.16) we obtain at leading order in ϕ,

$$\phi_t = D(1)\,\phi_{xx} + R'(1)\,\phi, \tag{10.2}$$

with $x > 0$ and $t \gg 1$. The form of (10.2) leads us to put

$$\phi(x,t) = t^{-\frac{1}{2}} \mathrm{e}^{R'(1)\,t}\{\phi_1(x) + \phi_2(x)\,t^{-1} + \ldots\} + O(\mathrm{e}^{2R'(1)\,t}) \tag{10.3}$$

as $t \to \infty$. After substitution into (10.2) and applying the boundary condition (2.18) at $x = 0$, we find that

$$\phi(x) = b, \quad \phi_2(x) = d - \tfrac{1}{4}bx^2/D(1), \tag{10.4}$$

where b and d are constants which cannot be determined in the asymptotic development as $t \to \infty$, and depend upon the details of $g(x)$ with $x = \sigma - O(1)$. Thus, from (10.1)–(10.4) we have

$$u(x,t) = 1 + bt^{-\frac{1}{2}} \mathrm{e}^{R'(1)\,t}\{1 + [\hat{d} - \tfrac{1}{4}x^2/D(1)]\,t^{-1} + \ldots\}, \tag{10.5a}$$

at $t \to \infty$ with $x = O(1)$, where $\hat{d} = d/b$. Clearly, this expansion develops a non-uniformity when $x = O(t^{\frac{1}{2}})$, and we require a further region.

Region II

In this region, $x = O(t^{\frac{1}{2}})$ as $t \to \infty$, and from (10.5a) $u = 1 + O[t^{-\frac{1}{2}} \mathrm{e}^{R'(1)\,t}]$. Therefore we introduce the scaled variable $\bar{x}$ as,

$$x = t^{\frac{1}{2}}\,\bar{x}, \tag{10.5b}$$

with $\bar{x} = O(1)$ as $t \to \infty$. We expand $u(\bar{x},t)$ in the form

$$u(\bar{x},t) = 1 + t^{-\frac{1}{2}} \mathrm{e}^{R'(1)\,t}\{G_0(\bar{x}) + t^{-1}G_1(\bar{x}) + \ldots\}, \tag{10.6}$$

as $t \to \infty$ with $\bar{x} = O(1)$. After substitution from (10.6) into equation (2.16) we obtain at leading order the following problem for G_0:

$$\left.\begin{aligned} &D(1)\,G_0'' + \tfrac{1}{2}\bar{x}G_0' + G_0 = 0, \quad 0 < \bar{x} < \infty, \\ &G_0(0) = b, \quad G_0(\bar{x}) \quad \text{bounded as} \quad \bar{x} \to \infty. \end{aligned}\right\} \tag{10.7}$$

The boundary conditions arise from matching with region I as $\bar{x} \to 0$, and to enable matching with region III as $\bar{x} \to \infty$. The solution to (10.7) is readily obtained as

$$G_0(\bar{x}) = b\,\mathrm{e}^{-\frac{1}{4}\bar{x}^2/D(1)},$$

and the expansion for $u(\bar{x},t)$ in this region is therefore,

$$u(\bar{x},t) = 1+t^{-\frac{1}{2}}\,e^{R'(1)\,t}\{b\,e^{-\frac{1}{4}\bar{x}^2/D(1)}+O(t^{-1})\}, \tag{10.8}$$

as $t\to\infty$ with $\bar{x} = O(1)$. The calculation of higher order terms in (10.8) reveals a further non-uniformity when $\bar{x} = O(t^{\frac{1}{2}})$, that is $x = O(t)$. This puts us into the wave-front region.

Region III

In this region $x \sim O(t)$ and the minimum speed permanent form wave-front appears at leading order. With the leading edge of the wave-front at $x = s(t)$, we introduce the coordinate y so that

$$x = s(t)+y, \tag{10.9}$$

with $y = O(1)$ and negative in this region. We expand $u(y,t)$ and $s(t)$ in the form

$$\left.\begin{aligned} u(y,t) &= u^*(y)+\chi(t)\,u_1(y)+\dots,\\ \dot{s}(t) &= v^*+\psi(t)\,v_1+\dots,\end{aligned}\right\} \tag{10.10}$$

as $t\to\infty$ with $y = O(1)$. Here $\chi(t)$, $\psi(t) = o(1)$ are (as yet) undetermined gauge functions, whilst $u^*(y)$ represents the minimum speed ($v = v^*$) permanent form travelling wave solution (see §4). After substituting from (10.9), equation (2.16) becomes

$$u_t-\dot{s}(t)\,u_y = D'(u)\,u_y^2+D(u)\,u_{yy}+R(u), \quad -\infty < y < 0, \tag{10.11}$$

which is to be solved subject to conditions (2.19a, b), which become

$$u(0,t) = 0, \quad u_y(0,t) = -\dot{s}(t), \tag{10.12}$$

together with matching to region II as $y\to-\infty$. On substituting from (10.10) into the edge conditions (8.12) we find that $\psi(t) = O(\chi(t))$. Therefore, without loss of generality, we put

$$\psi(t) \equiv \chi(t). \tag{10.13}$$

Furthermore, the matching of expansion (10.8) to expansion (10.10) in region III requires

$$\chi(t) = t^{-\frac{1}{2}}\,e^{[R'(1)-\frac{1}{4}v^{*2}/D(1)]\,t}. \tag{10.14}$$

With $\psi(t)$ and $\chi(t)$ given by (10.13), (10.14), we substitute from (10.10) into (10.11) and (10.12). At leading order we obtain the following linear boundary value problem for $u_1(y)$,

$$\begin{aligned} &D(u^*)\,u_1''+\{2D'(u^*)\,u^{*\prime}+v^*\}\,u_1'+\{R'(u^*)+D'(u^*)\,u^{*\prime\prime}\\ &\quad+D''(u^*)\,(u^{*\prime})^2-R'(1)+\tfrac{1}{4}v^{*2}/D(1)\}\,u_1 = -v_1\,u^{*\prime}, \quad -\infty < y < 0, \end{aligned} \tag{10.15}$$

$$u_1(0) = 0, \quad u_1'(0) = -v_1, \tag{10.16}$$

$$u_1(y) \sim b\,e^{-\frac{1}{2}v^*y/D(1)} \quad \text{as} \quad y\to-\infty. \tag{10.17}$$

Here prime $=$ d/dy and condition (10.17) is determined by matching to region II. The problem (10.15)–(10.17) is an eigenvalue problem for v_1. For a given $b > 0$, it can be shown that (10.15)–(10.17) has a unique solution. However, our primary interest is in determining v_1, and this can be achieved without a detailed solution of (10.15)–(10.17). Let $\gamma(y)$ be the solution of equation (10.15) and conditions (10.16) with $v_1 = 1$, and condition (10.17) replaced by $u_1(y) \sim O(e^{-\frac{1}{2}v^*y/D(1)})$ as $y\to-\infty$. Then,

$$\gamma(y) \sim \gamma_0\,e^{-\frac{1}{2}v^*y/2D(1)} \quad \text{as} \quad y\to-\infty, \tag{10.18}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

for some fixed constant γ_0. The solution to (10.15)–(10.17) is then readily obtained as

$$u_1(y) = v_1\,\gamma(y). \tag{10.19}$$

Moreover, from (10.18) we have

$$u_1(y) \sim \gamma_0\,v_1\,\mathrm{e}^{-\frac{1}{2}v^* y/D(1)} \quad \text{as} \quad y \to -\infty, \tag{10.20}$$

which, on comparing with condition (10.17) leads to

$$v_1 = \gamma_0^{-1} b. \tag{10.21}$$

Note that the constant b arose in region I, and is not determined in this long time asymptotic development, with it being a remnant of the earlier time evolution in the main support domain.

Finally, from (10.2), (10.14), (10.13) and (10.10) we find that the long time asymptotic wave-front propagation speed is given by

$$\dot{s}(t) \sim v^* + \gamma_0^{-1}\,bt^{-\frac{1}{2}}\exp\{[R'(1) - (\tfrac{1}{4}v_*^2/D(1))]\,t\} + \ldots,$$

and the minimum speed, permanent form, travelling wave is approached through exponentially small terms as $t \to \infty$. This rapid contraction onto the permanent form wave is evident in the numerical solutions of §9 (see figures 7–8).

11. Conclusions

We have considered piecewise classical solutions to the integral conservation law (2.1) subject to conditions (2.4)–(2.6). The problem has been reformulated into examining the initial boundary-value problem IBVP. The initial data $u_0\,g(x)$ has compact support with the support being $I_\sigma = [0, \sigma]$, and $g(x)$ analytic and monotonically decreasing in I_σ. In particular $g(x) = O([x-\sigma]^m)$ as $x \to \sigma^-$ for some $m = 0, 1, 2, \ldots$.

The existence of a family of permanent form travelling wave solutions (TW) to the integral conservation law (2.1) has been established, parametrized by their propagation speed $v \geqslant v^*$. For $v = v^*$ the TW has semi-infinite support, whereas for $v > v^*$ each TW has infinite support. For all initial data, we have shown that $u(x,t)$ (the solution to IBVP) develops into the TW with minimum speed $v = v^*$. In particular, as $t \to \infty$, the edge of the support domain $(0 \leqslant x \leqslant s(t))$ has

$$\dot{s}(t) \sim v^* + O[t^{-\frac{1}{2}}\,\mathrm{e}^{-ct}],$$

where $c = \frac{1}{4}v^{*2}D(1)^{-1} - R'(1)$, and the contraction onto the minimum speed TW is rapid, through terms exponentially small in t as $t \to \infty$.

For initial data with $m = 0, 1$ the edge of the support domain initiates motion immediately, at $t = 0$, with

$$\dot{s}(t) \sim \begin{cases} |g_1| + o(1), & m = 1, \\ O(t^{-\frac{1}{2}}), & m = 0, \end{cases}$$

as $t \to 0^+$. The cases $m = 1, 0$ give rise to impulsive and singular initial motion respectively. However, for initial data with $m \geqslant 2$, a waiting time $t_\omega > 0$ appears, and the edge of the support domain remains stationary until $t \to t_\omega^-$. Specific initial data in this class can always be chosen to set the waiting time to be as large or small as desired. In particular we have

$$t_\omega \geqslant \log(1 + \tfrac{1}{6}\lambda_0^{-1}),$$

where $\lambda_0 = \sup_{0 \leqslant x \leqslant \sigma}\{u_0 g(x)(x-\sigma)^{-2}\}$. For $m = 2$ we have

$$t_\omega \leqslant \log(1+\tfrac{1}{6}g_2^{-1}),$$

and so for initial data with $\lambda_0 = g_2$, then

$$t_\omega = \log(1+\tfrac{1}{6}g_2^{-1}).$$

When waiting times are present ($m \geqslant 2$) the initiation of motion of the edge of the support domain has been discussed. For $m = 2$ we find that as $t \to t_\omega^+$,

$$\dot{s}(t) \sim c_0 \quad \text{when} \quad t_\omega < \log(1+\tfrac{1}{6}g_2^{-1}),$$

for some constant $c_0 > 0$ (depending upon $g(x)$), whilst

$$\dot{s}(t) \sim \begin{cases} O([t-t_\omega]^{\frac{4}{3}}), & g_3 \neq 0 \\ O([t-t_\omega]), & g_3 = 0 \end{cases}$$

when $t_\omega = \log(1+\tfrac{1}{6}g_2^{-1})$. For $m \geqslant 3$, the initiation of motion of the edge of the support domain always has the form $\dot{s}(t) \sim c_0$ as $t \to t_\omega^+$, with $c_0 > 0$ depending upon $g(x)$.

For cases $m \geqslant 3$ and $m = 2$ with $t_\omega < \log(1+\tfrac{1}{6}g_2^{-1})$, the initiation of motion of the edge of the support domain is caused by global effects throughout the support domain on $0 \leqslant t < t_\omega$, whereas for $m = 2$ with $t_\omega = \log(1+\tfrac{1}{6}g_2^{-1})$, the initiation of motion is controlled purely by local conditions at the edge of the support domain.

Appendix

Here we demonstrate that the solution $u(x,t) \equiv 0$ to the initial boundary value problem (2.20)–(2.23) is unique. Let $u = \hat{u}(x,t)$, $s(t) \leqslant x < \infty$, $t \geqslant 0$ be a solution of (2.20)–(2.23). Then, following the approach adopted in the proof of proposition (3.1), we can readily establish that

$$0 \leqslant \hat{u}(x,t) \leqslant 1, \tag{A 1}$$

on $x \geqslant s(t)$, $t \geqslant 0$. We next integrate both sides of equation (2.20) with respect to x, over the interval $s(t) \leqslant x < \infty$. Since $\hat{u}(x,t)$ is classical in $x > s(t)$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{s(t)}^{\infty} \hat{u}(x,t)\,\mathrm{d}x\right\} + \hat{u}(s(t),t)\,\dot{s} = [D(\hat{u})\,\hat{u}_x]_{s(t)}^{\infty} + \int_{s(t)}^{\infty} R(\hat{u})\,\mathrm{d}x. \tag{A 2}$$

On using boundary conditions (2.22) and (2.23*a*, *b*), (A 2) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{s(t)}^{\infty} \hat{u}(x,t)\,\mathrm{d}x\right\} = \int_{s(t)}^{\infty} R(\hat{u})\,\mathrm{d}x. \tag{A 3}$$

On using (2.3*b*) and (A 1) in (A 3) we obtain

$$\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t} - \hat{\psi} \leqslant 0, \quad t > 0. \tag{A 4}$$

where
$$\hat{\psi}(t) = \int_{s(t)}^{\infty} \hat{u}(x,t)\,\mathrm{d}x, \quad t \geqslant 0. \tag{A 5}$$

In addition the initial condition (2.21) requires that

$$\hat{\psi}(0) = 0. \tag{A 6}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The differential inequality (A 4) may be re-written as, $(\hat{\psi}\,\mathrm{e}^{-t})_t \leqslant 0$, $t > 0$, which (after applying $\int_0^t \ldots \mathrm{d}t$ to both sides and using condition (A 6) leads to $\hat{\psi}(t) \leqslant 0$. However, (A 1) and (A 5) imply $\hat{\psi}(t) \geqslant 0$. Hence $\hat{\psi}(t) \equiv 0$, $t \geqslant 0$. Inequality (A 1) and (A 5) then lead to,

$$\hat{u}(x,t) \equiv 0, \quad x \geqslant s(t), \quad t \geqslant 0,$$

as required.

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